Interpretation of Dispersion Relation for Bounded Systems

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ABSTRACT

A treatment is given of the problem of constructing normal modes for an arbitrarily bounded system from roots of the linear dispersion relation $D(\omega, \underset{\sim}{k}) = 0$ for the corresponding infinite or periodically bounded system. For a system described by continuous macroscopic variables, and of general cylindrical form (uniform along an axis z, say), each transverse eigenmode gives rise to a set of axial normal modes constructed from a pair of dominant roots $k_z^{\pm}(\omega)$ of $D = 0$ satisfying the boundary conditions which are characterized by complex reflection coefficients for the dominant waves. The implications of the results for the interpretation of experiments on plasma waves and instabilities on finite cylinders is discussed, with particular reference to the effects of end-plate damping and axial current on Q-machines.

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I. INTRODUCTION

Linear perturbation analyses of uniform plasmas (or any continuous medium) are commonly carried out in the context of an initial value problem in rectangular coordinates for an infinite, i.e. unbounded, system. The perturbations are fourier transformed in space and either fourier transformed or more correctly, to take account of causality, Laplace transformed in time to derive a dispersion relation $D(\omega,\underset{\sim}{k},C_\gamma) = 0$ for perturbations proportional to $\exp i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{r})^\dagger$, where the C's represent steady state system parameters. Inhomogeneous plasmas also are commonly discussed for infinite rectangular geometry, with the gradients along x (say), and in the local approximation $k_x \gg C^{-1}\,\partial C/\partial x$ (i.e. weak inhomogeneity), so that again the perturbations can be fourier transformed in space.

The dispersion relation $D(\omega,\underset{\sim}{k}) = 0$ is regarded as giving the dependent, generally complex, variable ω in terms of the independent continuous real variable $\underset{\sim}{k}$, and wave types (α) are classified in terms of the various branches $\omega_\alpha(\underset{\sim}{k}$ real). An essential feature of this approach is that the modes $\omega(\underset{\sim}{k}$ real), for the same α and different $\underset{\sim}{k}$'s, as well as for different α's, are linearly independent. It follows that at long times the system is dominated by the mode giving the root $\omega(\underset{\sim}{k}$ real) with the minimum imaginary part $\omega_{i\ min}$. This corresponds to the slowest decaying mode for a stable system $(\omega_{i\ min} \geq 0)$, or the fastest growing mode for an unstable system $(\omega_{i\ min} < 0)$. Furthermore, theories for the nonlinear evolution of instabilities are also often developed[1,2] for unbounded systems in terms of these linearly independent modes ω(k real), leading to a homogeneous turbulent state which asymptotically is independent of the initial conditions, provided these have a reasonably smooth k-spectrum.

Sometimes, as in the original quasilinear treatment[3] of the weak bump-in-the-tail electron instability, the problem is formulated for a system of finite length, L, with periodic boundary conditions, and the

† Of the four possible conventions $\exp i\,(\pm\omega t \pm \underset{\sim}{k}\cdot\underset{\sim}{r})$ we choose this form for consistency with the work of Derfler and Briggs.

perturbations are fourier analyzed over a discrete set of linearly independent modes $k_n = 2n\pi/L$. However, in evaluating the total effect of nonlinear wave-particle (or wave-wave) interaction, the sums are replaced by integrals. Effectively then, the system length is allowed to tend to infinity, so the discrete modes become a continuum, and the result is independent of L.

Since all physical systems are necessarily bounded, but not in general periodically, the important question arises of the relevance of theories for infinite, or periodically bounded, systems to the behavior of real systems. Sometimes, in adapting such theories to practical geometries, periodic boundary conditions are appropriate where a coordinate closes on itself. For instance, drift modes derived in rectangular coordinates (x,y,z) are adapted to cylindrical geometry (r,θ,z) by identifying k_y with m/a for waves varying as $\exp -im\theta$, where m is the azimuthal mode number and a is the radius at which the mode is localized. Again, in adapting solutions for a cylinder to toroidal geometry, k_z is identified with n/R where n is the toroidal mode number and R is the major radius. However, periodic boundary conditions are not always appropriate. In particular, for the important practical case of a cylindrical system of finite length, periodic boundary conditions can never be justified on physical grounds since they involve a mathematical assumption concerning how the system is continued beyond the boundaries, a question devoid of physical meaning. The implications of periodic boundaries are brought out very clearly in computer simulations, where particles reaching a boundary are reintroduced at the opposite boundary instantaneously and with the same velocity. Clearly this is a condition which can never be realized in practice.

Of course, the reason why theories are usually derived for infinite, or periodically bounded, systems is perfectly clear. It is precisely to avoid the complicated questions involved in realistically modelling the behavior of fields and particles at boundaries, and also to avoid the necessity of solving an eigen problem for some particular finite geometry. In this way one arrives at an ideal theory which describes the essential behavior of the medium, uncomplicated by the effects of boundary conditions or finite geometry. However, if such ideal theories are to be of more than just theoretical interest, and have utility for understanding or predicting the behavior of real systems, then the introduction of boundaries must not modify the results in any significant manner. Under some conditions perhaps, the ideal theory may give a reasonable representation of the behavior of real systems, but the conditions under which this is true are difficult to specify. In many cases, however, the ideal theory clearly does not give a good representation of the real system, as the following considerations show.

Firstly, boundaries can support surface waves[4-6] which are not contained in the "infinite" dispersion relation, but which must sometimes be included to satisfy boundary conditions. Secondly, when the boundaries are not periodic, the perturbations cannot be fourier analyzed into a set of linearly independent modes. To be more explicit, if one fourier analyzes the perturbations over the finite length, then the separate fourier components are in general coupled by the boundaries, and the normal modes consist of infinite sums of these components. Thirdly, when the boundaries are not periodic, linear instabilities generally grow spatially instead of, or as well as, temporally, and the resulting nonlinear (turbulent) state is in general inhomogeneous. In the case that a steadily oscillating state occurs, the perturbations are better described in terms of the roots of $D = 0$ for complex $\underset{\sim}{k}$ and real ω. This approach is particularly relevant for externally driven systems as discussed by Self[7] in connection with beam-plasma instabilities and by the Stanford group[8] in connection with low frequency waves and instabilities on a magnetized positive column. In this case the root of $D = 0$ giving the maximum spatial growth for real ω is more significant than that giving the maximum temporal growth for real $\underset{\sim}{k}$.

It should be noted that in general there is no one-to-one correspondence between the branches $\underset{\sim}{k}_{\beta}$ (ω real) and ω_{α} ($\underset{\sim}{k}$ real), and one arrives at quite different, but equally valid, classifications of wave types on the two bases. Only for simple propagating waves (ω and $\underset{\sim}{k}$ both real) does a correspondence exist; more generally a connection can only be made through a process of conformal mapping in which ω and $\underset{\sim}{k}$

are regarded as complex variables. The very different picture which emerges according as one treats $\underset{\sim}{k}$ or ω as real is exemplified by the work of Gould[9] for ion waves and by Derfler[10] for electron waves in one-dimensional collisionless Maxwellian plasmas, and by Self[11] for ion and drift waves and instabilities in weakly ionized inhomogeneous magnetoplasmas.

When a dispersion relation is nearly satisfied by purely real $(\omega,\underset{\sim}{k})$ in some vicinity $(\omega_r,\underset{\sim}{k}_r)$ say, so that $D(\omega_r + i\omega_i,\underset{\sim}{k}_r) = D(\omega_r,\underset{\sim}{k}_r + i\underset{\sim}{k}_i) = 0$ with (ω_i/ω_r), $(|\underset{\sim}{k}_i|/|\underset{\sim}{k}_r|) \ll 1$, the spatial and temporal growth or decay rates in this vicinity are related by $\omega_i = -\underset{\sim}{k}_i \cdot \nabla_k \omega|_{\omega_r,\underset{\sim}{k}_r}$. This applies, for instance, for the unstable waves in the interaction of a weak electron beam with a plasma[12] when the beam is hot (resonant case) but not when the beam is cold (non-resonant case). Drummond[13] recalculated the quasilinear theory of the weak bump-in-the-tail electron instability for the case of steady spatial growth in a half-space which, unlike the initial value problem in infinite geometry he treated earlier, corresponds to a physically realizable situation, if one neglects the effect of the second boundary. Apart from some weak time-average quasi-potential effects associated with the inhomogeneous electric fields, he found a quasilinear relaxation in space similar to the relaxation in time given by the initial value problem. The spatial and temporal relaxation scales are related by the group velocity of the unstable waves $\nabla_k \omega|_{\omega_r,\underset{\sim}{k}_r}$.

This is an example of how the ideal theory can be interpreted to yield answers relevant to a real situation. A similar example is the interpretation of the ideal theory of homogeneous turbulence in ordinary fluids[14] to describe the inhomogeneous turbulence produced by a grid in a fluid stream, where, in that case, the temporal and spatial scales are related by the stream velocity. However, this congruence between the initial value problem in infinite geometry $(t \geq 0,\ -\infty < z < \infty)$ and the steady problem for a half-space $(z \geq 0,\ -\infty < t < \infty)$ does not generally hold. For instance for the non-resonant interaction of a cold weak beam with a plasma,[12] the relative linear spatial growth rate $|\underset{\sim}{k}_i|/|\underset{\sim}{k}_r|$ may be large even though the relative linear temporal growth rate (ω_i/ω_r) is small, and such a congruence does not exist.

More generally one can pose the question of whether it is possible, and if so under what conditions, to construct from roots of the "infinite" dispersion relation solutions describing the behavior of bounded systems. Clearly, if one can do this, even under some restricted conditions, it greatly enhances the utility of the "infinite" theory and avoids the necessity of treating each bounded system as a separate problem. A related question is that of constructing solutions to describe externally driven systems. Here it may be noted that, strictly, it only makes sense to discuss an externally driven problem for a system having a boundary, since one can only apply an external source there.

These questions are of great practical significance for the design and interpretation of laboratory plasma experiments, which may be broadly divided into two categories. In the first, one is concerned with devising experiments to verify the dispersion relation of stable or unstable waves under controlled conditions. Here one tries to make the geometry and boundary conditions as simple as possible to facilitate theoretical interpretation. For this purpose the favorite vehicle has been a long cylindrical plasma either of the internally generated type (dc or rf positive columns, PIG discharge etc.) or of the externally generated type (surface ionization, duoplasmatron, hollow cathode or rf sources). With care a uniform plasma can be created which is effectively one-dimensional i.e. the steady-state parameters are a function only of radius. In the second category, one is concerned with the instabilities of plasmas occurring in various fusion-type devices, where geometrical simplicity is lost by the need to introduce mirror, multipole or shear magnetic fields, toroidal geometry, etc., in order to contain a hot, dense plasma. Here the emphasis is more on suppressing or controlling instabilities than studying them per se.

In either category of experiment, when the system is unstable, and self-excited instabilities grow in time to some nonlinearly saturated state, one is obliged, in the absence of a nonlinear theory, to compare the characteristics, in particular the frequency and wavenumber spectrum, of the nonlinear state with linear theory. Clearly, no such comparison is strictly possible, though the assumption is usually made that the frequencies and wavenumbers having the highest temporal growth rates (for real $\underset{\sim}{k}$) in the linear theory will be most in evidence in the nonlinear regime. Such a comparison is most plausible when there is an external parameter C which can be adjusted to take the system across a boundary C_0 from stability to instability, since for values of C just above C_0 it is to be expected that the nonlinear state will most closely reflect the predictions of linear theory. Sometimes a direct check of the linear temporal growth rate can be made either by suddenly switching C across the boundary value C_0,[15] or by feedback stabilizing the instability and suddenly switching off the feedback.[16]

A feature common to many experimental studies of temporally growing instabilities is that the results are interpreted in terms of the roots $\omega(\underset{\sim}{k}$ real) of the "infinite" dispersion relation without enquiring properly into the effects of boundaries. Values of real $\underset{\sim}{k}$ determined by the system dimensions are employed even though the boundaries are clearly not periodic. In fact, there seems to be a rather widespread misconception, no doubt engendered by the theorist's predilection for infinite or periodic systems, that the behavior of real systems must always be interpreted (or interpretable) in terms of the roots of the "infinite" dispersion relation for real $\underset{\sim}{k}$.

There is, however, another method for studying instabilities under linear conditions which avoids this difficulty. It is applicable when it can be arranged that the instability grows in space rather than in time, and is essentially the same as is commonly employed for studying the dispersion of stable waves. The system is externally excited at some location at some (real) frequency ω_0 and the amplitude and phase of the resulting waves is measured as a function of position. Thus the complex propagation constant $(\underset{\sim}{k}_r + i\underset{\sim}{k}_i)$ of the dominant wave is compared with the roots $\underset{\sim}{k}(\omega$ real) of the linear dispersion relation, and as C exceeds C_0 a transition from an attenuating to an amplifying wave is observed. Whether the instability grows to a nonlinear level depends on the product of the spatial growth rate and the system length, together with the excitation amplitude. This method facilitates a rather thorough check of the linear dispersion relation and has been employed for studying beam-plasma[7,17] and positive column instabilities.[8,18]

In this paper we consider the question posed above, of whether and how it is possible to construct, from roots of the "infinite" dispersion relation, solutions describing the behavior of bounded systems. We especially consider cylindrical systems of finite length and have in mind the application to basic wave and instability studies where this geometry is so frequently employed. Because of the existence of surface waves, it must be concluded that in general the answer to our question must be negative, which leads to the discouraging conclusion that for rigorous results each and every bounded problem must perforce be treated individually. However, one is left with the feeling that it ought to be possible, under some conditions at least, to draw conclusions about the behavior of bounded systems from a knowledge of the dispersion relation for the infinite system. After all, in simple cases, such as acoustic or electromagnetic waves in passive lossless media, there is no difficulty in interpreting the roots of the "infinite" dispersion relation to describe the propagation of externally excited waves, or in using such roots to construct normal mode solutions for cavity resonators with ideal boundaries.

However, the extension of these ideas to active (unstable) media and more general boundary conditions is not entirely trivial, but may be made, subject to certain restrictions, as we discuss. To do this two steps are necessary. Firstly, in Sec. II, we make use of the existing theory for the interpretation of waves and instabilities in infinite systems, making an elementary extension of that theory. Secondly, in Sec. III, we borrow from transmission line theory the concept of characterizing a boundary by a complex impedance or reflection coefficient to construct normal modes for an arbitrarily bounded system. This allows us to discuss, in general terms, how the system behavior depends on the form of the "infinite" dispersion relation and the terminations. These ideas are applied to collisional instabilities in a Q-machine in Sec. IV. The paper is concluded by a discussion in Sec. V.

II. INTERPRETATION OF DISPERSION RELATIONS FOR INFINITE SYSTEMS

If one derives a linear dispersion relation $D(\omega,\underset{\sim}{k}) = 0$ simply by assuming that the perturbed variables are proportional to $\exp i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r})$, there arises a difficulty in interpreting its roots with complex $\underset{\sim}{k}$ because one cannot distinguish, purely from the sign of $\underset{\sim}{k}_i$, whether the wave is amplifying or attenuating, since one does not know whether to consider increasing or decreasing position values. [For complex ω roots no difficulty arises because we always consider time increasing]. For stable systems there is no problem because on physical grounds all such waves must be attenuating, but for active systems there is an ambiguity.

Historically, such difficulties were apparently first encountered in connection with the theory of distributed electron tubes of the travelling wave type. There the question was resolved by a combination of calculations for specific systems together with a strong element of physical intuition. Similar problems arose in the theory of plasma streaming instabilities[19] in connection with noise radiation in solar radio bursts. Pierce[20] and Twiss[21] emphasized that the difficulties arose because the problems were improperly formulated, there being no explicit reference to initial and boundary conditions. In a very detailed study of propagation in electron-ion streams, by the method of Laplace transforms, Twiss[21] showed how to distinguish amplifying and attenuating waves. He also gave the first indication of the distinction between convective and absolute (non-convective) instabilities in the response of an unstable system to a localized initial disturbance (Fig. 1). The latter distinction was also made by Landau and Liftshitz[22] and brought out very clearly by Sturrock.[23] For the special case of the coupling of two simple propagating modes, Sturrock showed how to distinguish, purely from the topology of the conformal mappings of $D(\omega,\underset{\sim}{k}) = 0$, between amplifying and attenuating waves, and between convective and absolute instabilities, emphasizing the essential similarity between convective instabilities and amplifying waves.

Subsequently a number of workers[24-35] have developed these ideas in varying degrees of generality and have given criteria for distinguishing

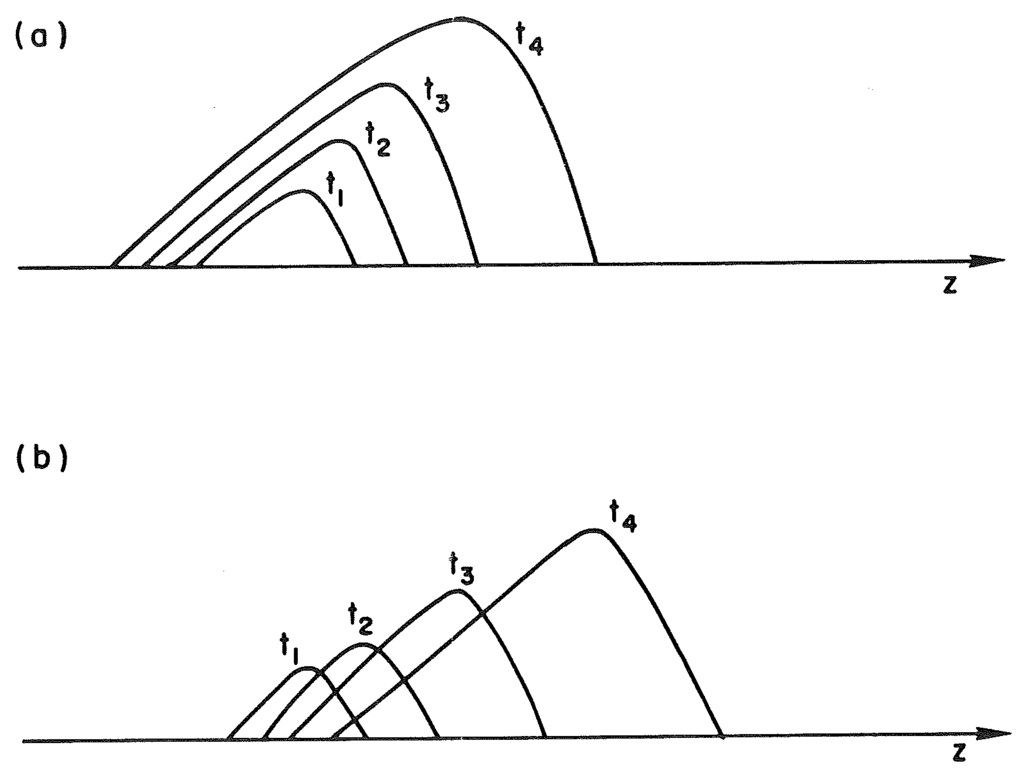

FIG. 1. DEVELOPMENT OF A LOCALIZED PERTURBATION AS A FUNCTION OF TIME ($t_1 < t_2 < t_3 < t_4$) FOR (a) ABSOLUTE INSTABILITY AND (b) CONVECTIVE INSTABILITY.

convective from absolute instabilities and amplifying from attenuating waves. While these criteria are stated in different ways, and vary in their ease of application, they mostly appear to have the same essential content and yield the same results, except perhaps in exceptional cases. Most authors consider a one-dimensional system, but Dysthe[31] gives results for three dimensions. Some take an infinite system and use fourier transforms while others take a semi-infinite system and use Laplace transforms in space. Many treatments are restricted to certain types of dispersion relations, in particular ones that are algebraic or polynomials in ω and k. Some, especially Derfler,[24] discuss more general cases, in particular when $D(\omega,\underset{\sim}{k})$ is a multi-valued function as arises, in general, in the kinetic theory of hot plasmas. In the following, we briefly outline the theory, following for the most part the treatment of Briggs,[28] and for simplicity consider the case when $D(\omega,\underset{\sim}{k})$ is a (finite) polynomial (of order α in ω and β in $\underset{\sim}{k}$) as is usually the case in fluid treatments of plasmas.

(A) Classification of Instability and Wave Types

Consider a time-invariant system of general cylindrical form, infinite and uniform along z, for which a linear perturbation analysis by the ansatz $\exp i(\omega t-kz)$ yields a dispersion relation $D(\omega,k) = 0$. It is implied that the transverse eigen problem has been solved and that, for simplicity, we are considering excitation in one of the linearly independent transverse eigenmodes. The roots $\omega_\alpha(k \text{ real})$ of $D = 0$ give the temporal growth or decay rates for an initial perturbation in the form of an infinite sinusoid $\exp -ikz$, where we assume that $\omega_i(k)$ is finite for all real k. By definition the system is unstable if for any real k there exists a root with $\omega_i < 0$; otherwise, i.e. if $\omega_{i\,\min} \geq 0$ for all real k, the system is stable.

Now consider an initially quiescent system excited by a source $s(t,z) \equiv g(z)\, f(t)$ localized to $|z| \leq d$ and switched on at $t = 0$, i.e. $g(z) = 0$ for $|z| > d$ and $f = 0$ for $t < 0$. The system response is written

$$\psi(t,z) = \int_{-\infty}^{\infty} \int_{-\infty+j\sigma_0}^{+\infty+j\sigma_0} G(\omega,k)\ f(\omega)\ g\ (k)\ \exp\ i(\omega t-kz)\ \frac{d\omega dk}{(2\pi)^2}\ . \qquad (1)$$

Here ψ represents any first order variable, $f(\omega)$ is the Laplace transform of $f(t)$, $g(k)$ is the fourier transform of $g(z)$ and $G(\omega,k) = [D(\omega,k)]^{-1}$ is the Laplace-fourier transform of the Greens function. The fourier integral path (F.I.P.) is taken along the real k axis while the Laplace integral path (L.I.P.) is taken below all singularities of the integrand, i.e. σ_0 sufficiently negative. Interchanging the order of integration, the response can be written

$$\psi(t,z) = \int_{-\infty+i\sigma_0}^{+\infty+j\sigma_0} F(\omega,z)\ f(\omega)\ \exp\ i\ \omega t\ \frac{d\omega}{2\pi}\ , \qquad (2)$$

where

$$F(\omega,z) = \int_{-\infty}^{\infty} G(\omega,k)\ g(k)\ \exp\ -ikz\ \frac{dk}{2\pi}\ . \qquad (3)$$

Consider first the evaluation of F for any $\omega = \omega_L$ on the L.I.P. Since $g(z)$ is localized, then for any physically realizable form, $g(k)$ is an entire function (having no singularities in the finite k-plane). Thus the only singularities in the integrand of (3) are those of $G(\omega_L,k)$. In the simple cases to which the present discussion is limited, these singularities of G will be poles at the zeros of $D(\omega_L,k) = 0$. The integral, Eq. (3), can then be evaluated, by closing the contour at infinity in the upper half k-plane for $z < -d$ and in the lower half k-plane for $z > d$, as a sum of the residues of the poles of $G(\omega_L,k)$ at $k = k_\beta^-(\omega_L)$ and $k_\beta^+(\omega_L)$ in the upper and lower half-planes respectively:

$$F(\omega_L, z < -d) = \sum_\beta \frac{ig(k_\beta^-)\ \exp -ik_\beta^- z}{\left[\frac{\partial D}{\partial k}(\omega_L, k)\right]_{k=k_\beta^-}} \tag{4a}$$

and

$$F(\omega_L, z > d) = \sum_\beta \frac{-ig(k_\beta^+)\ \exp -ik_\beta^+ z}{\left[\frac{\partial D}{\partial k}(\omega_L, k)\right]_{k=k_\beta^+}} \quad . \tag{4b}$$

Here we have accounted only for simple poles. In general higher order poles only occur for discrete ω_L at isolated positions in the k-plane and may be treated as a merging of simple poles as discussed later. It may be noted that the F's in Eqs. (4) are given as a sum of modes which for any ω_L on the L.I.P. decay away from the source. Moreover, it is determined whether a given root $k_\beta(\omega_L)$ appears in the response for $z < -d$ or $z > d$.

Now consider the Laplace inversion, Eq. (2). To find the asymptotic $(t \to \infty)$ response we deform the L.I.P. as far as possible into the upper half plane, when it is clear that the asymptotic response is governed by the lowest singularity of the integrand $F(\omega,z)\ f(\omega)$ in the ω-plane.

At this point we specialize to an (undriven) initial value problem, taking $f(t) = \delta(t)$, a delta function, so that $f(\omega) = 1$, and we are only concerned with the singularities of $F(\omega_L,z)$. It is now clear that in this case it is sufficient that the L.I.P. be taken below the lowest branch ω_α(k real) of $D = 0$,i.e. $\sigma_0 < \omega_{i\ min}$ (k real). As ω traverses the L.I.P., the poles of $G(\omega_L,k)$ trace contours in the k-plane, namely the contours $\omega = \omega_r + i\sigma_0$ in the map of ω into the k-plane via $D = 0$ (Fig. 2a). Further, as the L.I.P. is raised in the ω-plane (e.g. $\sigma_0 \to \sigma_1$) such poles may cross the real k-axis. This will happen first when the L.I.P. is raised to intersect the lowest branch ω_α(k real) i.e. when $\sigma \geq \omega_{i\ min}$. When this happens, the function F must be redefined as its analytic continuation $\tilde{F}$ as the F.I.P. is deformed to continue to include the same poles as before.

In this way the L.I.P. may be continuously raised $(\sigma_1 \to \sigma_2)$ until two poles of $G(\omega_L,k)$ collide or merge through the F.I.P., pinching it

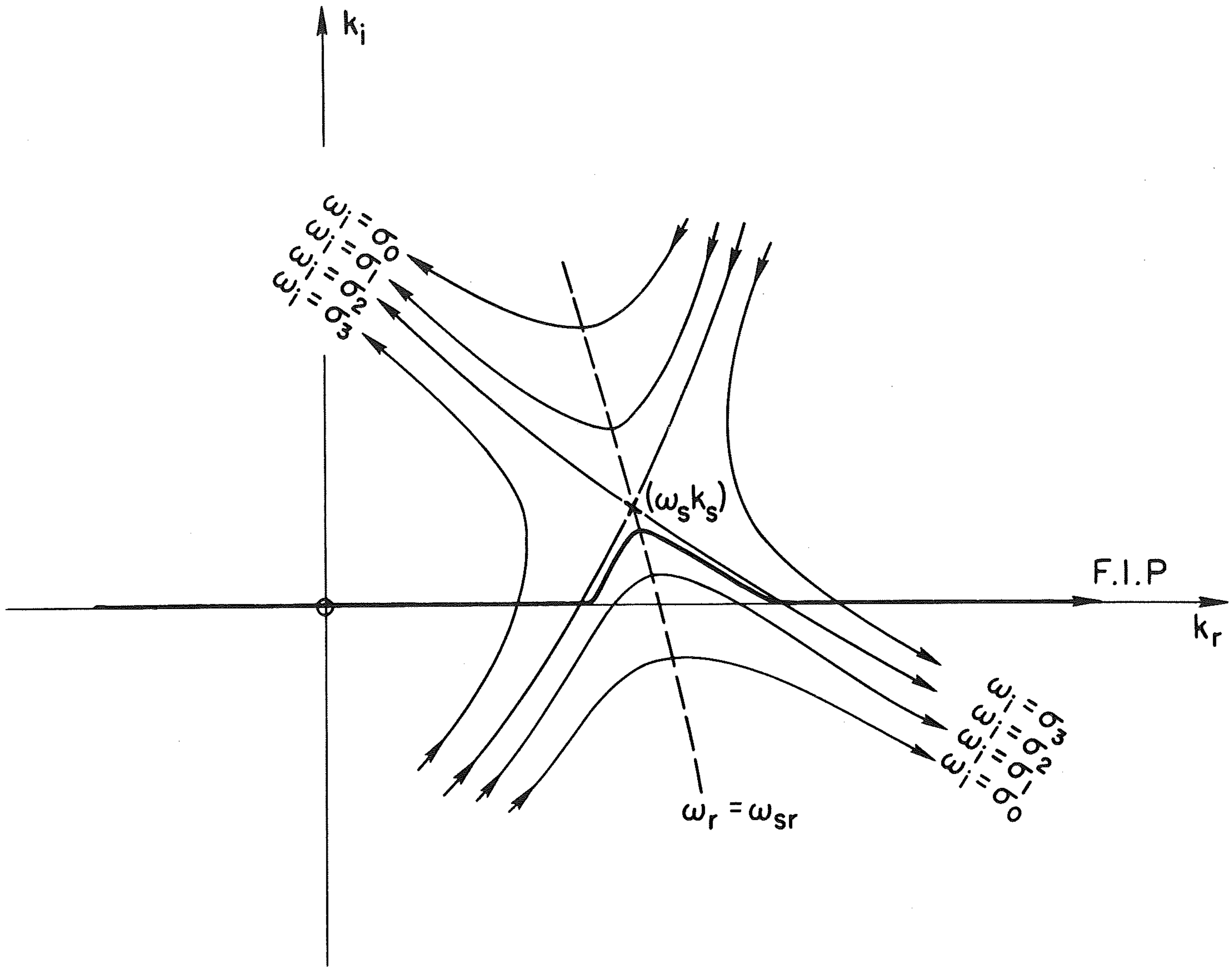

FIG. 2. (a) MAP OF ω INTO THE k-PLANE VIA $D = 0$ SHOWING THE FOURIER INTEGRAL PATH AND ITS DEFORMATION TO INCLUDE THE SAME POLES OF $G(\omega_L, k)$ AS THE L.I.P. IS RAISED. TWO POLES COLLIDE AT THE SADDLE POINT (ω_s, k_s) PINCHING THE F.I.P. BETWEEN THEM.

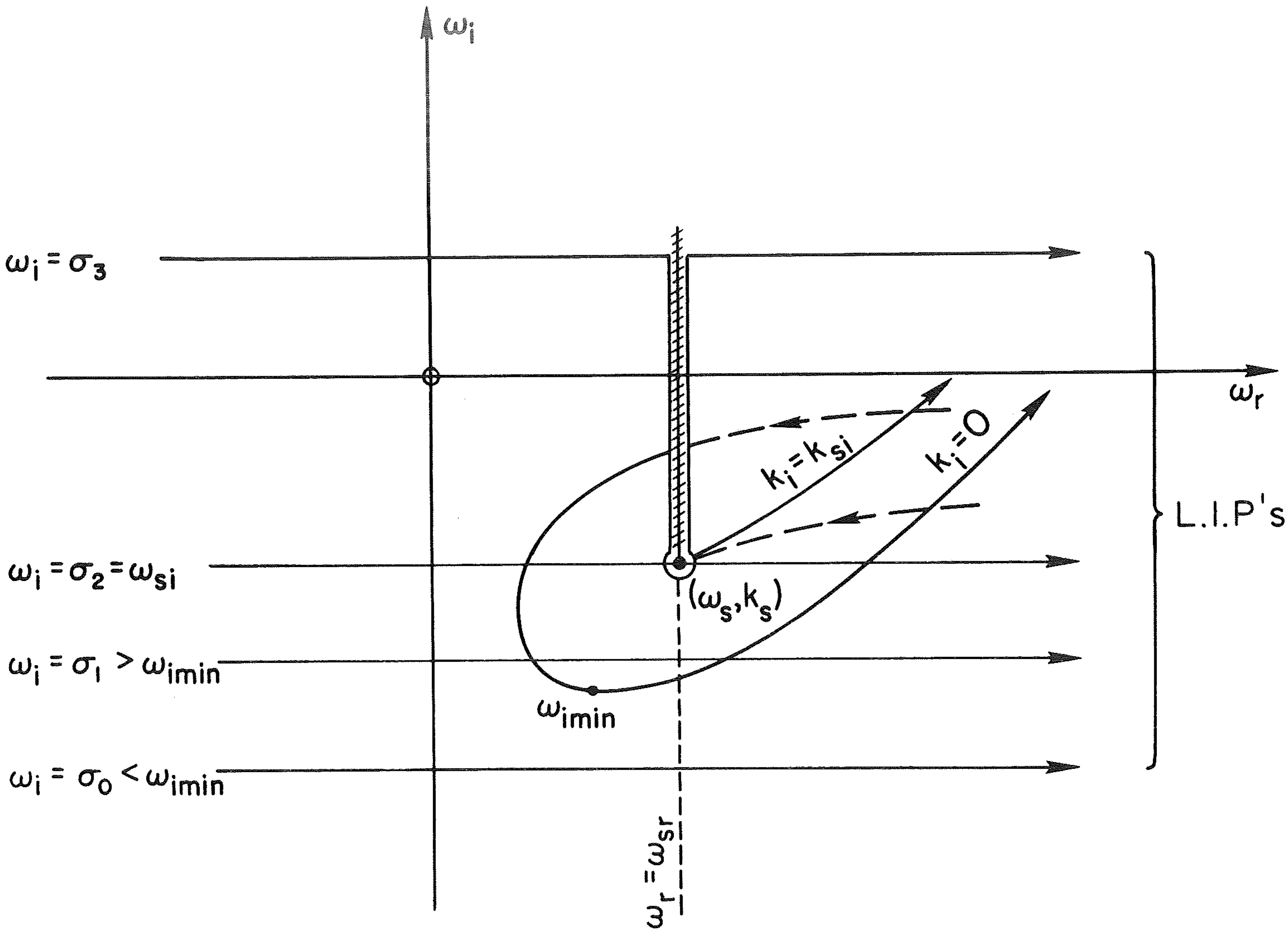

FIG. 2. (b) MAP OF k INTO THE ω-PLANE VIA $D = 0$ SHOWING THE REAL k-AXIS $(k_i = 0)$ THE BRANCH POINT (ω_s, k_s) AND VARIOUS LAPLACE INTEGRAL PATHS.

between them. When this happens at some (ω_s, k_s) we have a double root $k_s(\omega_s)$ of $D = 0$, i.e. a saddle point of $\omega(k)$ in the map of ω into the k-plane or, equivalently, a branch point of $k(\omega)$ in the map of k into the ω-plane via $D = 0$. In raising the L.I.P. still further $(\sigma_2 \to \sigma_3)$ it must be deformed around the branch cut of which the branch point (ω_s, k_s) forms one end point (Fig. 2b). The asymptotic response is clearly governed by the lowest such branch point in the ω-plane and can be evaluated as

$$\psi(z,t) \xrightarrow[t \to \infty]{} \frac{g\ (k_s)}{\left[2\pi i \left(\frac{\partial^2 D}{\partial k^2}\right)\left(\frac{\partial D}{\partial \omega}\right)\right]^{1/2}_{\omega_s, k_s}} \frac{\exp i(\omega_s t - k_s z)}{t^{1/2}} \tag{5}$$

If the lowest such branch point (ω_s, k_s) lies in the lower half plane $(\omega_{si} < 0)$ the response eventually increases without limit (in a linear analysis) at all (finite) z, as described by Eq. (5), and the system is by definition absolutely unstable. On the other hand, if the lowest such branch point lies in the upper half plane $(\omega_{si} > 0)$ the response eventually decays to zero at all finite z, according to Eq. (5). In this case, if the system is unstable it is convectively unstable and the perturbation grows but convects away. In all cases the asymptotic response has an exponential envelope in space (except near the front of the disturbance) given by $\exp -ik_{si}z$.

In general, saddle points of $\omega(k)$, i.e. branch points of $k(\omega)$, are given by the simultaneous solution of $D = 0$ and $\partial D/\partial k = 0$, and at such points the group velocity is zero $(d\omega/dk = 0)$. It should be emphasized that not all such saddle/branch points are relevant; only those branch/saddle points which correspond to a merging of roots $k(\omega)$ of $D = 0$ from opposite half k-planes as ω_i is increased from $-\infty$, and thus pinch the F.I.P. This is the basic phenomenon on which the various criteria and prescriptions for distinguishing convective and absolute instabilities are based. It may be noted that in general the lowest relevant branch/saddle point lies above the minimum of the lowest branch ω_α (k real), i.e. $\min_s \omega_{si} > \omega_{i\ \min}$ (k real). Exceptionally, the branch point can lie on the real k branch at its minimum, i.e. $\min_s \omega_{si} = \omega_{i\,min}$ (k real).

The asymptotic response to the impulse $g(z)\delta(t)$ is expressed in terms of these relevant points of zero group velocity through Eq. (5) and represent the natural or characteristic responses of the medium. Physically it is implied that a system which is absolutely unstable cannot exist in a steady state, since any perturbation, however small, will grow and eventually fill all space. On the other hand, if the system is not absolutely unstable (i.e. either stable or convectively unstable) the response will eventually decay at all finite z and the system can exist in a steady state.

We now consider the response of the system to a continuing perturbation (a driven system) at $z = 0$ by taking for the source $s(t,z) = \delta(z) \exp i\omega_0 t$, where the perturbation can be considered as externally imposed or due to internal thermal fluctuations. When ω_0 is real, it can be regarded as any fourier component of a periodic excitation (internal or external). Alternatively, when ω_0 is complex, it can be regarded as an external excitation of increasing or decreasing amplitude according to $(\exp - \omega_{0i} t) \exp i\omega_{0r} t$.

The process described above for evaluating the asymptotic response proceeds as before, but this time, in evaluating the Laplace integral, Eq. (2), we must take account of the singularity of $f(\omega) = [i(\omega - \omega_0)]^{-1}$, i.e. the pole at $\omega = \omega_0$. Clearly, if ω_0 lies above the lowest relevant branch point (ω_s, k_s) of $k(\omega)$, i.e. if $\omega_{0i} > \min_s \omega_{si}$, then the foregoing arguments are unchanged, and the asymptotic response is given by Eq. (5), as before. On the other hand, if ω_0 lies below the lowest relevant branch point i.e. if $\omega_{0i} < \min_s \omega_{si}$, then the asymptotic response will be governed by the drive and can be evaluated as

$$\psi(t,z) \xrightarrow[t \to \infty]{} \tilde{F}(\omega_0, z) \exp i\,\omega_0 t \,. \tag{6}$$

This is clearly expressed through Eq. (4), with $g = 1$, as a sum over normal modes $k_\beta(\omega_0)$. The analytic continuation, $F \to \tilde{F}$, determines whether a particular root appears in the response for $z < 0$ or $z > 0$, and hence, from the sign of k_i, whether it grows or decays in space away from the excitation plane $z = 0$.

Hitherto, in this type of analysis, ω_0 has been taken as real, corresponding to steady excitation, and the analysis is only applicable for systems which are not absolutely unstable. In this case the argument given above leads to criteria for deciding on which side of the excitation point the various waves $k_\beta(\omega_0)$ appear and hence, from the sign of k_i whether they are amplifying or damped waves. The extension to complex ω_0 has been made for purposes which will become clear when we consider bounded systems in Sec. III.

In the present context the physical meaning is clear. Even if the system is absolutely unstable, provided the drive increases more rapidly than the fastest growing absolute instability, the asymptotic response will be governed by the drive and not by the natural response of the system. On the other hand, if the system is not absolutely unstable (i.e. stable or convectively unstable), the asymptotic response will again be governed by the drive, even though it is a decreasing one, provided it decays less rapidly than the slowest decaying natural response of the system. In either case the criteria usually employed for real ω_0, for determining whether the roots $k_\beta(\omega_0)$ appear in the response for $z < 0$ or $z > 0$, may still be employed for complex ω_0.

(B) Application to Physical Systems

While the analysis of instability and wave types, outlined above, is very illuminating, it should be remembered that it is strictly applicable only to infinite systems, and it might therefore be thought to have no immediate application to practical, bounded systems. However, provided one assumes that the uniformity of the steady state is maintained in the presence of boundaries, the analysis leads to useful results in two respects.

Firstly, it is clear that the response of a bounded system to an initially localized perturbation is given by the "infinite" analysis up until such time as a leading edge of the perturbation reaches a boundary. Further, given proper boundary conditions on the waves, one could, in principle, follow the transient response as disturbances are reflected to and fro between the boundaries. This concept is, however, mainly of heuristic value; usually we do not require such a detailed description.

Secondly, a matter of more practical importance, is the fact that in some circumstances it is possible, experimentally, to realize effectively reflectionless terminations as in a transmission line matched load. This is achieved, for instance,[8] for ion acoustic and drift waves on a long discharge column where the "uniform" axial magnetic field is allowed to slowly diverge at the ends, and the electrodes are located well beyond the uniform field region.[†] In such cases one effectively simulates, as far as the central uniform region is concerned, an infinite system, and the results of the "infinite" analysis can be directly applied, with the following conclusions.

If the "infinite" dispersion relation reveals an absolute instability, then the finite system bounded by reflectionless terminations cannot exist in a steady state. Perturbations will grow, as described asymptotically by Eq. (5) until nonlinear effects become important. On the other hand, if the "infinite" dispersion relation reveals a convective instability, the system will exist in a stationary state, governed by the dominant amplifying wave i.e. that root $k_\beta(\omega \text{ real})$ with the largest spatial growth rate. In the absence of external excitation, this is driven by noise fluctuations and the amplitude is controlled by the frequency spectrum of the noise at the input end, i.e. the end from which waves grow most strongly. Depending on the input-end noise level, the spatial growth rate and the system length, the noise will be amplified to a level which may either remain in the linear regime or reach nonlinear saturation at some location. In the former case the system remains linear, even though it is unstable. In the latter case it is clear that the nonlinear (turbulent) state is inhomogeneous.

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† It is not always possible to simulate a reflectionless termination by gradually tapering the system ends. For instance for Langmuir waves, which only propagate for $\omega > \omega_p$, tapering off the plasma density produces a distributed reflection, however gradual the taper.

III. NORMAL MODES FOR A FINITE CYLINDER

In this section we discuss the construction of normal modes for a uniform cylindrical system of finite length and arbitrary terminations from appropriate roots of the linear dispersion relation for the equivalent infinite system.

The conditions under which it is valid to treat the finite system as a terminated length of the infinite system needs some discussion. Briefly, it is valid for ordinary dielectrics and cold plasmas and also, subject to conditions discussed later, for fluids and plasmas treated via fluid equations. It is, however, not in general valid for collisionless plasmas treated by kinetic equations. The reason for the distinction between dielectric or fluid treatments on the one hand, and kinetic treatments on the other, is that in the former the particle dynamics used to calculate the charge ρ and current $\underset{\sim}{J}$ in Maxwell's equations are determined purely by the local fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$, whereas in the latter ρ and $\underset{\sim}{J}$ are determined via integrals over the particle trajectories, including earlier encounters with the boundaries.[36-39] Thus in a cold or fluid model a boundary imposes conditions on the fields and fluid variables only at the boundary, whereas in a kinetic model the effect of the boundary conditions on the particles is felt on the particle dynamics throughout the system.

(A) Case when $D(\omega,k)$ is Quadratic in k

We take the same basic model as for the infinite cylindrical system of Sec. II, but of finite length $z_1 \le z \le z_2$. Initially, for simplicity, we consider the case where the dispersion relation $D(\omega,k) = 0$ for a given transverse eigenmode of the infinite system is quadratic in k.

Now consider the two roots $k(\omega)$ of $D = 0$, where ω is restricted to values below the lowest relevant branch/saddle point, i.e. $\omega_i < \min_s \omega_{si}$. These roots can then be interpreted as the waves excited on an infinite system by a localized external drive at (complex) frequency ω. We shall assume that they correspond to one (k^-) excited on the negative z side and one (k^+) excited on the positive z side as determined by whether they lie respectively in the upper or lower half k-planes for $\omega_i < \omega_{i\,min}$ (k real). This assumption is justified for most cases of interest but there are exceptional cases where it is not.†

Apart from the requirement that, for $\omega_i < \omega_{i\,min}$ (k real), $k_i^- > 0$ and $k_i^+ < 0$, which determines the labelling of the roots, there is no restriction on the signs of $k_r^\pm$ or $k_i^\pm$. If $k_i^- > 0$ and $k_i^+ < 0$, the corresponding waves are attenuating; with the opposite signs they are amplifying. The signs of $k_r^\pm$ determine the directions of the phase velocities $v_p^\pm \equiv \omega_r/k_r^\pm$. Since wave energy must flow away from the excitation point, the roots $k^\pm$ can be classified as forward or backward waves according as $v^\pm$ is directed away from or towards this point.

We now write the total solution for the bounded system as the sum of these two waves,

$$\psi = \psi^+ + \psi^- = \Psi^+ \exp i(\omega t - k^+ z) + \Psi^- \exp i(\omega t - k^- z) , \qquad (7)$$

where the complex amplitudes are written

$$\Psi^\pm = |\Psi^\pm| \exp i\phi^\pm .$$

The wave ψ^+ is regarded as excited at z_1 by reflection of the wave ψ^- and, likewise, ψ^- is excited at z_2 by reflection of ψ^+.

We define the (time-independent) complex reflection coefficients ρ_1, ρ_2 as the ratios of the amplitudes of the reflected to incident waves:

$$\rho_1 \equiv |\rho_1| \exp \pm i\theta_1 \equiv \frac{\psi^+(z_1)}{\psi^-(z_1)} = \frac{|\Psi^+|}{|\Psi^-|} \exp i[-(k^+ - k^-)z_1 + (\phi^+ - \phi^-)], \qquad (8a)$$

† For instance, for a one dimensional monoenergetic beam $[D \equiv 1-\omega_b^2/(\omega-kv_0)^2]$ or a cold beam-plasma system $[D \equiv 1-\omega_p^2/\omega^2 - \omega_b^2/(\omega-kv_0)^2]$, the two roots correspond to waves both excited on the same (downstream) side.

$$\rho_2 \equiv |\rho_2| \exp \pm i\theta_2 \equiv \frac{\psi^-(z_2)}{\psi^+(z_2)} = \frac{|\Psi^-|}{|\Psi^+|} \exp i[(k^+ - k^-)z_2 - (\phi^+ - \phi^-)], \quad (8b)$$

where $0 \leq |\rho_{1,2}| \leq 1$ and $-\pi < \theta_{1,2} \leq \pi$.

Here we have assumed that the phase of the reflected wave is given, either from physical arguments or by experiment, as leading or lagging the incident wave by a certain angle $|\theta|$, and adopt the convention that $\theta > 0$ when the reflected wave leads and $\theta < 0$ when it lags. In order to preserve this convention, irrespective of the sign of ω_r , the alternate signs are introduced in Eqs. (8), and should be taken as + or - according as $\omega_r \gtrless 0$. [See Fig. 3.]

It is assumed that the terminations are such that the boundary conditions can be met by these two waves in a single transverse eigen mode (together perhaps with surface waves which are localized to the boundaries, as discussed later). This will be true if the terminations are uniform in the transverse plane. More generally the terminations can couple different transverse modes together, as for example in magnetron anodes where straps and output terminations are deliberately used to create a desired spectrum of coupled transverse modes.[40] The theory given here can be generalized to such cases in a straightforward manner. It is also assumed that the reflection coefficients are frequency independent, though again the analysis can be carried through when they are specified functions of frequency.

Elimination of Ψ^+/Ψ^- between Eqs. (8) yields

$$(k^+ - k^-)L = [2n\pi \pm (\theta_1 + \theta_2)] - i[\log|\rho_i|\ |\rho_2|] , \quad (9)$$

where $L = (z_2 - z_1)$ is the system length, $n = 0, \pm 1, \pm 2$ etc, and the term $(\theta_1 + \theta_2)$ takes the same sign as ω_r . Thus specifying the reflection coefficients ρ_1, ρ_2 together with the system length quantizes the difference between the roots $k^+(\omega)$, $k^-(\omega)$ of the infinite dispersion relation to a discrete set of (generally complex) values.

The physical content of Eq. (9) is just that the total phase shift around the loop must be an integral multiple of 2π , and that the loop

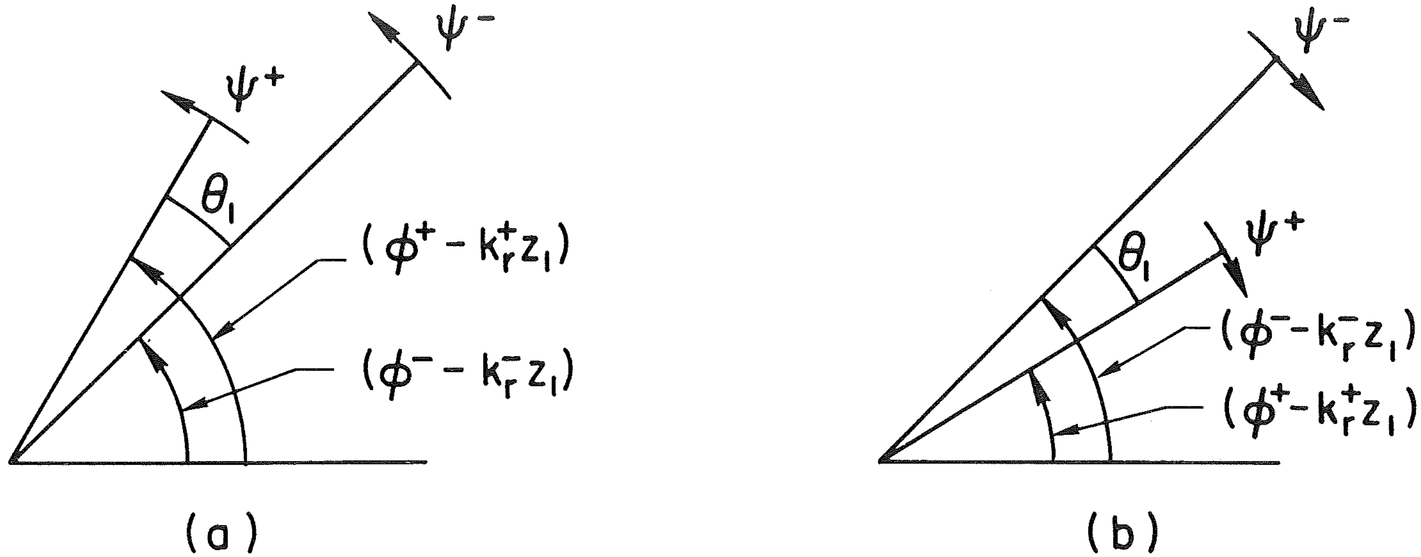

FIG. 3. PHASOR REPRESENTATION OF INCIDENT WAVE ψ^- AND REFLECTED WAVE ψ^+ AT ($z = z_1$, $t = 0$), SHOWING ψ^+ LEADING ψ^- BY θ. IN (a) $\omega_r > 0$ AND THE PHASORS ROTATE IN THE POSITIVE (ANTICLOCKWISE) DIRECTION, WHILE IN (b) $\omega_r < 0$ AND THE PHASORS ROTATE CLOCKWISE.

gain must be unity. Since $\log |\rho_1|\ |\rho_2| \leq 0$, and, for $\omega_i < \omega_{i\ min}$(k real), $(k_i^+ - k_i^-) < 0$, it is clear that Eq. (9) can only be satisfied for $\omega_i \geq \omega_{i\ min}$(k real). In general, for $\rho_{1,2} \neq 1$, one of the waves (at least) must be amplifying. For perfect reflections $\rho_1 = \rho_2 = 1$, the spatial growth rate of the amplifying wave equals the spatial damping rate of the attenuated wave. Exceptionally, for reciprocal systems (D even in k), Eq. (9) may be satisfied by purely real $k^{\pm}$, for perfect reflections.

We may now determine the (complex) frequencies ω_n of these axial eigen modes which satisfy both Eq. (9) and the "infinite" dispersion relation. Writing the latter in the form

$$a(\omega)k^2 + b(\omega)k + c(\omega) = 0 \tag{10}$$

we have $(k^+ - k^-) = \pm[b^2(\omega) - 4a(\omega)c(\omega)]^{1/2}/a(\omega)$. (11)

The frequencies ω_n of the axial modes are then given by eliminating $(k^+ - k^-)$ between Eqs. (9) and (11) and solving for ω . Hence the normal mode frequencies are given by

$$F(\omega,\rho_1,\rho_2,L,n) \equiv a^2(\omega)\left\{[2n\pi \pm (\theta_1+\theta_2)] - i\ \ln |\rho_1|\ |\rho_2|\right\}^2 L^{-2}$$

$$- b^2(\omega) + 4a(\omega)c(\omega) = 0 \quad . \tag{12}$$

In case the system is reciprocal, i.e. D is even in k , then $k^+ = -k^-$ and Eq. (12) reduces to

$$F(\omega,\rho_1,\rho_2,L,n) \rightarrow a(\omega)\left\{[2n\pi \pm (\theta_1+\theta_2)] - i\ \ln |\rho_1|\ |\rho_2|\right\}^2 L^{-2}$$

$$+ 4c(\omega) = 0 \quad . \tag{12a}$$

In Eq. (12) the whole range of n values (n = 0, ±1, ±2 etc.) are allowed, but with the + sign on $(\theta_1+\theta_2)$ only roots with $\omega_r > 0$ are admitted, while with the - sign only roots with $\omega_r < 0$ are admitted. Even with this limitation, Eq. (12) may still give extraneous roots because it was derived by squaring Eqs. (9) and (11) to remove the sign ambiguity in Eq. (11). To distinguish these, $D = 0$ (i.e. Eq.(10)) must be solved for each admitted root ω of Eq. (12) to give k^+ and k^- separately, identifying them by the prescription that $k_i^+ < 0$ and $k_i^- > 0$ as $\omega_i \to -\infty$. Only those roots giving $(k^+ - k^-)$ with the correct sign to satisfy Eq. (9) are proper roots ω_n.

If $D(\omega,k)$ is of order α in ω, then $F(\omega)$ is of order 2α or less. Because of the possibility of extraneous roots it does not seem possible in general to determine the number α' of proper roots ω_n for fixed ρ_1, ρ_2, L, n, except that $\alpha' \le 2\alpha$.

It should be emphasized that unless a root has $\omega_{n\alpha' i} < \min_s \omega_{si}$ the above procedure breaks down, because the initial restriction $\omega_i < \min_s \omega_{si}$, which allows one to identify the two roots $k(\omega)$ as waves excited on either side of an excitation point, is violated. We shall use the term axial eigenmode to denote only those roots which have $\omega_{n\alpha' i} < \min_s \omega_{si}$. It is clear that these modes must have $\omega_{n\alpha' i} \ge \min_\alpha \omega_{\alpha i}$ (k real), because otherwise both roots $k_n^{\pm}$ represent attenuated waves and Eq. (9) cannot be satisfied.

Some of the implications of the foregoing analysis can be summarized as follows:

(i) If $\min_{n\alpha'} \omega_{n\alpha' i} > \min_s \omega_{si}$, there are no axial eigenmodes in the sense used here.

(ii) If $\min_{n\alpha'} \omega_{n\alpha' i} < \min_s \omega_{si}$, the asymptotic time response will be governed by this lowest mode and will grow or decay as $\exp -(\min_{n\alpha'} \omega_{n\alpha'_i})t$.

(iii) As either or both of the terminations are made more lossy (i.e. $|\rho_1||\rho_2|$ made smaller) the spatial growth rates of the component waves $k_n^{\pm}$ must increase and the $\omega_{n\alpha'}$ rise higher in the ω-plane, so that for sufficiently lossy terminations condition (i) must apply, unless of course there are no relevant branch/saddle points ω_s. The latter condition applies for instance for a passive lossless waveguide, or Bohm and Gross waves $(D(\omega,k) \equiv \omega^2 - \omega_0^2 - k^2c^2)$ in which case there are always decaying axial eigenmodes however lossy the terminations, except in the limit $|\rho_1||\rho_2| = 0$.

(iv) When $\min_\alpha \omega_{\alpha i}$ (k real) < 0 but $\min_s \omega_{si} > 0$, so that the infinite system is convectively unstable, then in general, for sufficiently lossless reflections, there will be growing axial eigenmodes. However, if the values of reflection coefficients and system length combine to make the allowed values of $k_n^\pm$ fall outside the range where unstable waves occur, there will be no growing modes. This may happen, for instance, when the range of real k giving temporal growth is restricted to small values (long wavelengths) and the system is too short to contain them. In this case, or when the reflections are made sufficiently lossy to stabilize the growing modes, then the system can exist in a steady state. The system behavior can then be described in terms of the roots $k^\pm(\omega$ real) of $D = 0$, representing waves excited internally by noise or externally at the boundaries, taking account of the excitation of reflected waves. Although in this case linear theory indicates a steady state, it may happen that the waves excited (by noise or externally) at boundaries grow to a nonlinear level in the system length.

(v) When $\min_\alpha \omega_{\alpha_i}$ (k real) < 0 and $\min_s \omega_{si} < 0$, so that the infinite system is absolutely unstable, then, except in the special case when ω_s also happens to be the lowest point of the real k contour, there will in general be growing modes for sufficiently reflecting terminations. Of course it may happen that the values of ρ_1, ρ_2, L combine to disallow any growing modes as described in (iv). As $|\rho_1||\rho_2|$ is reduced from unity the growth rates of the axial eigenmodes decrease until $\min_{n\alpha'} \omega_{n\alpha' i} > \min_s \omega_{si}$, when there are no longer any axial eigenmodes. To describe the system now, it seems it would be necessary to follow the development of an initial perturbation as it grows and spreads and is successively reflected from the boundaries. However, it is clear that perturbations must grow indefinitely since even in the limit $\rho_1 = \rho_2 = 0$, when the system is effectively unbounded, one knows that perturbations described by Eq. (5) will grow indefinitely. Hence one gets the result that while convective instabilities can always be stabilized, i.e. prevented from growing in time, by sufficiently lossy boundaries, absolute instabilities cannot.

The movement of the growing mode frequencies ω_n as $|\rho_1||\rho_2|$ is decreased from unity, keeping θ_1, θ_2 fixed, as described in (iv) and (v) above, is illustrated in Fig. 4 for a case when $D(\omega,k)$ is even in k. In this case the real k contour folds on itself and the relevant branch/saddle point occurs for $k = 0$. The mapping of k into the ω plane consists of two sheets but, because the roots $k^{\pm}(\omega)$ occur in pairs $k^+ = -k^-$, the two sheets have been represented in one diagram. Also the cases of convective and absolute instability are represented in the same diagram by changing the position of the real ω axis.

For perfect reflections, $|\rho_1| = |\rho_2| = 1$, the modes ω_n lie on the real k contour and are pure standing waves formed by superposition of real $k^{\pm}$ waves, so there is no average power flow. Three such modes $\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}$ are shown. As $|\rho_1||\rho_2|$ is decreased, the frequencies ω_n migrate towards the upper half plane; (e.g. $\omega_n^{(1)}$ in Fig. 4) decreasing the growth rate. The modes are no longer pure standing waves, but are composed of two oppositely travelling amplifying waves. Part of the energy released by the unstable medium flows into the lossy ends, the rest going to give the temporal increase in wave energy. For the case of convective instability, when all the modes have migrated into the upper half plane (e.g. $\omega_n^{(2)}$ in Fig. 4) the power flow to the ends is more than the unstable medium can supply so the modes decay in time and the system reverts to a steady state. For the absolutely unstable case, when $|\rho_1||\rho_2|$ is small enough that all the ω_n have migrated to positions above the branch point ω_s (e.g. $\omega_n^{(3)}$ in Fig. 4) there are no longer any axial eigen modes.

More specific examples are discussed in Sec. IV.

(B) Case when $D(\omega,k)$ is of Higher Order in k

When the dispersion relation is of order $\beta > 2$ in k, the various roots $k(\omega)$ can all be identified, for $\omega_i < \min_s \omega_{si}$, as waves excited on the + or - sides of an excitation point. Usually β will be even and there will be pairs of roots $k_\beta^{\pm}(\omega)$ corresponding to waves excited on either side for each wave type. For instance, for a two component plasma treated from the first two moment equations, $\beta = 4$ and there is a pair of pressure waves associated primarily with the electrons

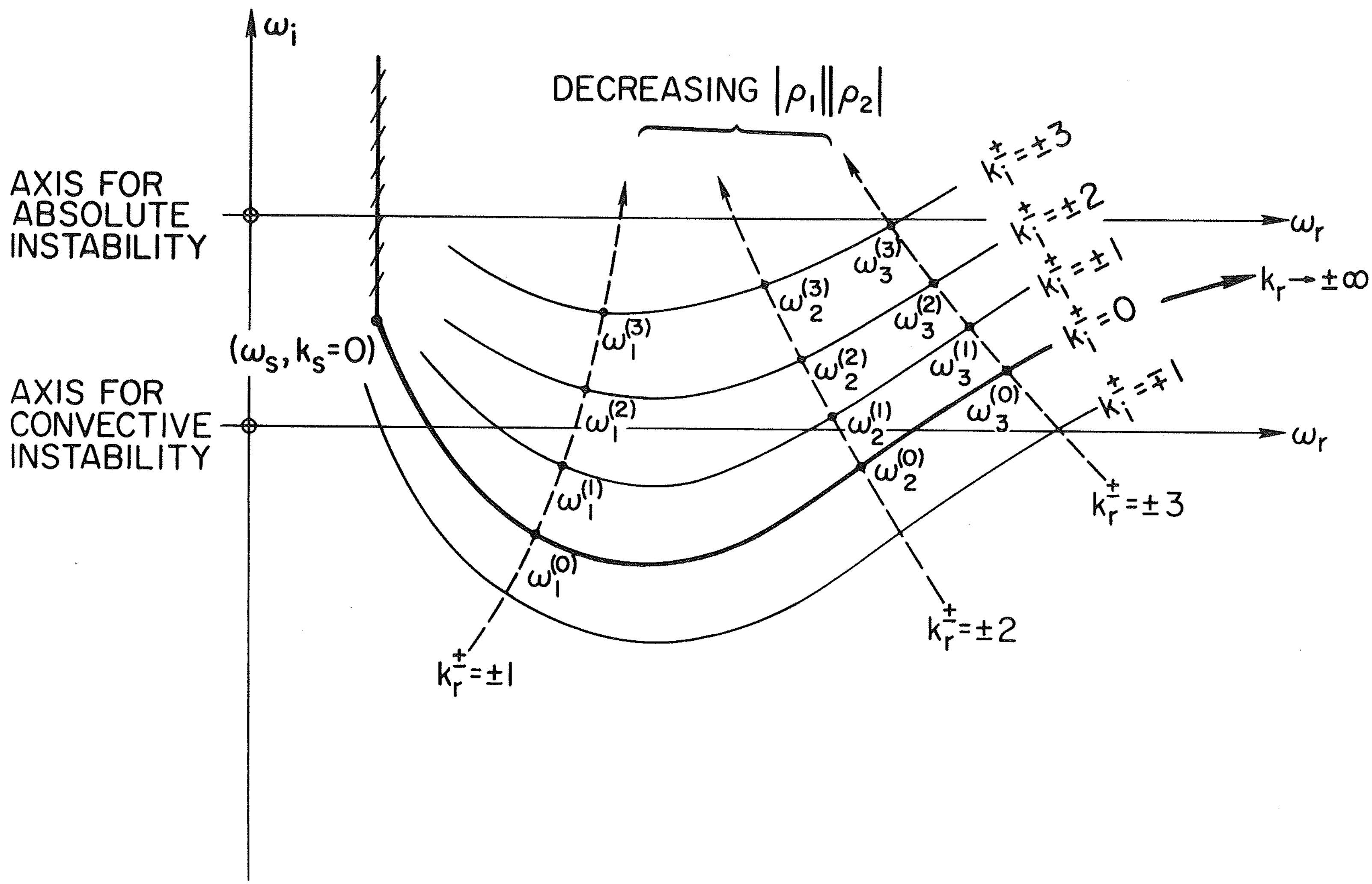

FIG. 4. MAP OF k INTO THE ω-PLANE FOR A CASE WHEN $D = 0$ IS EVEN AND QUADRATIC IN k. THE TWO SHEETS ARE REPRESENTED BY $\pm$ SIGNS. THE NORMAL MODES ω_n ARE SHOWN MIGRATING UPWARDS AS THE REFLECTIONS BECOME MORE LOSSY. THE CASES OF CONVECTIVE AND ABSOLUTE INSTABILITY ARE INDICATED BY THE ALTERNATIVE REAL ω-AXES.

(Langmuir waves) and another pair associated primarily with the ions (ion waves). In such a case we can write the total solution as

$$\psi = \psi_e^- + \psi_e^+ + \psi_i^- + \psi_i^+$$

where

$$\psi_{e,i}^{\pm} = \Psi_{e,i}^{\pm} \exp i(\omega t - k_{e,i}^{\pm} z) \tag{13}$$

We should then need to define a set of reflection coefficients $\rho_{ee}, \rho_{ii}, \rho_{ei}, \rho_{ie}$ for each termination to take account of the fact that a wave of a given type can excite by reflection not only the same type but also the other type. In principle one can eliminate the Ψ's to obtain a relation among the four wave vectors $k_{e,i}^{\pm}$ in terms of the ρ's which serves to determine the mode frequencies from the dispersion relation. In practice, however, the procedure is hardly tractable.

Fortunately, the cases of most practical interest correspond to the situation where one pair of waves is dominant and the others are strongly attenuated and excited only very locally to the terminations. Over most of the system only the dominant waves have appreciable amplitude and may be amplifying or only weakly attenuated.

The situation is analogous to that in ordinary electromagnetic waveguides[41,42] used in a frequency range where only the lowest transverse eigenmode propagates freely. Evanescent higher order transverse eigenmodes are excited locally to the termination and the impedance or reflection coefficient for the dominant mode is measured at a point sufficiently far from the termination that the evanescent modes are negligible. The locally excited higher modes, which are necessary to satisfy the boundary conditions in a full treatment of the problem via Maxwell's equations, contribute to the reflection coefficient for the dominant waves.

In the present case we can lump together the effects of the evanescent modes together with those of surface waves, if any, into the reflection coefficient of the dominant wave. The theory of the previous section is then applicable as far as Eq. (9) which determines the differences $(k^+ - k^-)_n$ for the dominant waves. The only problem that arises is to solve the dispersion relation for the mode frequencies ω_n for the allowed $(k^+ - k^-)_n$ values. It is clear that the problem is determined but it is difficult to give a general method when $\beta > 2$. In practice, however, there should be no difficulty in solving particular problems either graphically or numerically.

(C) Axial Mode Structure

In the simple case when the system is reciprocal, and the terminations are purely reactive $(|\rho_{1,2}| = 1)$, one constructs an axial eigen mode by superposition of two travelling waves of equal amplitude and real k's of opposite sign to obtain a pure standing wave of the form

$$(\exp i\omega_n t) \cos (k_n z + \chi_n) \quad .$$

More generally, when the terminations are lossy $(|\rho_{1,2}| < 1)$ and, or, the system is non-reciprocal, the modes will be composed of separate waves with k^+ and k^- complex, and the total solution is a partial standing wave.

It is useful to express the total solution in the form

$$\psi_n(t,z) = \Psi_n(z) \cos [\omega_{nr} t + \chi_n(z)] \exp - \omega_{ni} t \tag{14}$$

where Ψ_n and χ_n are real functions of z, because in practice it is the amplitude Ψ_n and phase χ_n of the total wave which can be measured as functions of z. Moreover, a measurement of $\Psi(z)$ and $\chi(z)$ allows one to determine the reflection coefficient as in a transmission line impedance measurement.

From Eqs. (8) we have

$$\frac{|\Psi^+|}{|\Psi^-|} = |\rho_1|^{z_2/L} \, |\rho_2|^{z_1/L}$$

and

$$(\phi^+ - \phi^-) = n\pi \pm (\theta_1 - \theta_2)/2 + (k_r^+ - k_r^-)(z_1 + z_2)/2 \ .$$

The values of the relative amplitude and phase difference of the component waves depends on the origin of the z-coordinate. For simplicity and without loss of generality, we may take the origin at the center so that $(z_1 + z_2) = 0$, $z_1 = -L/2$, $z_2 = + L/2$. Then we have:

$$\frac{|\Psi^+|}{|\Psi^-|} = \frac{|\rho_1|^{1/2}}{|\rho_2|^{1/2}} \tag{15a}$$

and

$$(\phi^+ - \phi^-) = n\pi \pm (\theta_1 - \theta_2)/2 . \tag{15b}$$

Since we are not interested in the absolute phase, it is convenient to take

$$\phi^\pm = \Phi \pm \phi$$

where

$$2\phi = (\phi^+ - \phi^-) = n\pi \pm (\theta_1 - \theta_2)/2 \tag{16}$$

is the phase difference between the component waves at the center, and the sign is taken to be the same as that of ω_r .

Then, to within an arbitrary amplitude factor and an arbitrary time phase, the normal mode can be written in the form (14) where

$$[\Psi(z)]^2 = \left\{ \begin{array}{l} |\rho_1| \ \exp 2\, k_i^+ z + |\rho_2| \ \exp 2\, k_i^- \, z \\ + 2\, |\rho_1|^{1/2} |\rho_2|^{1/2} \exp\, (k_i^+ + k_i^-)\ z \cos\, [2\phi - (k_r^+ - k_r^-)\ z] \end{array} \right\} \tag{17}$$

and

$$\tan \chi(z) = \frac{[|\rho_1|^{1/2} \exp k_i^+ \ z \sin\, (\phi - k_r^+ z) - |\rho_2|^{1/2} \exp k_i^- \ z \sin\, (\phi + k_r^- z)]}{[|\rho_1|^{1/2} \exp k_i^+ \ z \cos\, (\phi - k_r^+ z) + |\rho_2|^{1/2} \exp k_i^- \ z \cos\, (\phi + k_r^- z)]} . \tag{18}$$

In Eqs. (17), (18), the quantities ϕ, $(k_r^+ - k_r^-)$ are determined in terms of $\rho_{1,2}$, L, n from Eqs. (16), and (9) respectively. In general, when the medium is non-reciprocal, the values of $k_r^\pm$, $k_i^\pm$ depend on the form of the "infinite" dispersion relation and must be found by the method of the previous section. Hence, in general, the axial mode pattern depends not only on the system length and terminations, but also on the medium.

However, when the medium is reciprocal we have

$$k_i^+ L = - k_i^- L = \frac{1}{2} \ln |\rho_1||\rho_2| = k_i L \text{ (say)}$$

and

$$k_r^+ L = - k_r^- L = [n\pi \pm (\theta_1 + \theta_2)/2] = k_r L \text{ (say)} .$$

Then

$$[\Psi(z)]^2 \to |\rho_1| \exp 2 k_i z + |\rho_2| \exp - 2k_i z + 2|\rho_1|^{1/2} |\rho_2|^{1/2} \cos 2(\phi - k_r z) \qquad (17a)$$

and

$$\tan \chi(z) \to \frac{[|\rho_1|^{1/2}\exp k_i z - |\rho_2|^{1/2}\exp - k_i z] \tan (\phi - k_r z)}{[|\rho_1|^{1/2}\exp k_i z + |\rho_2|^{1/2}\exp - k_i z]} \quad . \qquad (18a)$$

Since k_r, k_i are determined when ρ_1, ρ_2, L, n are specified, the mode pattern is determined purely by the system length and terminations, and is independent of the form of the infinite dispersion relation, when the latter is even in k . When, also, the terminations are identical, $|\rho_1| = |\rho_2| = |\rho|$ and $\theta_1 = \theta_2 = \theta$, so that $\Phi = n\pi/2$, we have:

$$[\Psi(z)]^2 \to [\cosh 2 k_i z + \cos(n\pi - 2k_r z)]$$

$$\to \cosh [2(\ln |\rho|)z/L] + \cos [2(n\pi \pm \theta) z/L - n\pi] \qquad (17b)$$

and

$$\tan \chi(z) \to \tanh k_i z \tan (\frac{n\pi}{2} - k_r z)$$

$$\to - \tanh k_i z \tan [(n\pi \pm \theta) z/L - n\pi/2] \quad . \qquad (18b)$$

If, further the terminations are purely reactive, $|\rho_1| = |\rho_2| = 1$, we have $k_i = 0$ and

$\Psi(z) \to \cos [(n\pi \pm \theta)z/L - n\pi/2]$,

$\chi(z) \to 0$.

In this case the mode patterns are pure standing waves with cos or sin form about the center according as n is even or odd.

(D) Determination of Reflection Coefficient

The construction of axial eigen modes for a finite length system from pairs of roots $k^{\pm}(\omega)$ of the infinite system dispersion relation, as described above, depends upon the characterization of the terminations by complex reflection coefficients for these waves. This enabled us to describe the finite system behavior in general terms; in particular the stabilizing effect of end losses and the partial standing wave structure of the modes for systems with end loss and/or a non-reciprocal dispersion relation. However, if the theory is to have utility for specific systems, one must have a method for determining the reflection coefficient either by calculation or measurement.

Unfortunately, in many plasma experiments using discharge columns, the boundaries are ill-defined and the boundary conditions difficult to specify in terms precise enough for calculations. For a Q-machine, for which there are approximate theories[43,44] for the physical processes at the hot end-plates, the situation is better. However, since there are sheaths at the end-plates, it is strictly not possible to regard the plasma between the plates as a section of a uniform infinite system. There is also a more fundamental problem in using a fluid description for a bounded system, for example collisional drift waves in a Q-machine, in that a fluid description breaks down close to a boundary. When the magnetic field is normal to the boundary the fluid equations fail within a mean free path, λ, of the boundary, while if B is parallel to the surface it fails within a distance λ or the gyro-radius, ρ, whichever is the smaller. Consequently, it is difficult, in general, to calculate the reflection coefficients for the dominant waves $k^{\pm}(\omega)$ from a knowledge of the physical processes at the boundaries.

For these reasons it is more satisfactory to regard the reflection coefficient as an experimentally determined quantity. The effect of sheaths and the failure of the fluid description are lumped together with those of possible localized surface waves and higher modes (for $\beta > 2$) into an effective reflection coefficient measured at distances from the boundary greater than the mean free path (or gyro radius) and sheath thickness. This standpoint is analogous to transmission line practice where, although the reflection coefficient can be calculated in principle for arbitrary terminations, in practice it is usually much simpler to measure it.

Except when $D(\omega,k) = 0$ indicates absolute instability and also there are no normal axial modes, it is possible to measure the reflection coefficient either by the usual transmission line method or by a modification thereof. If it is practicable to effect a more or less matched termination at one end, $|\rho_1| \approx 0$, so that any convective instability is stabilized, it is possible to determine ρ_2 as a function of ω by externally exciting a wave and measuring the amplitude and phase of the resulting partial standing wave in the standard fashion.[42] If this is not practicable then one has to deal with the resonant modes and extract the reflection coefficients from a comparison of the measured amplitude and phase with the theoretical expressions of Eqs. (17) and (18). If the system is stable, either because $D(\omega,k) = 0$ indicates stability, or because a convective instability is stabilized by end loss, then the resonances would be externally excited. On the other hand, if the system is unstable one would analyze the standing wave pattern of the self-excited instability. This would probably only be possible for parameter values a little above threshold when a single mode is weakly excited. In either case the reflection coefficient can be measured as a function of ω by varying the system length.

IV. APPLICATION TO COLLISIONAL DRIFT WAVES IN Q-MACHINES

The Q-machine, as it has been developed, represents the best defined and understood experimental vehicle for the study of waves and instabilities in highly-ionized plasmas. Much of the early work was confused due to an inadequate understanding of the steady state. With the advent of von Goeler's[43] steady-state theory, which emphasized the importance of end-plate recombination, and its extension by Chen,[44-45] who pointed out the dominant rôle of end-plate temperature inhomogeneity on the plasma potential profile and radial diffusion, more closely controlled conditions and more consistent results have been obtained.

The Q-machine has been used particularly for studying density gradient driven collisional drift waves and instabilities, and a reasonably detailed correspondence between theory and experiment has been established.[46,47] An important factor in this achievement was the decoupling of density gradient drift waves from edge oscillations associated with the strong radial temperature and potential gradients near the hot-plate edge. This was obtained by restricting the neutral flux to the central region of the hot-plate where the temperature gradient is small.

(A) Collisional Drift Wave Theories

The various theories for collisional drift waves are derived in rectangular geometry (x, y, z) with the magnetic field B along z and the zero order gradients of n (and T) along x, and in the local approximation $k_x \gg \chi \equiv (1/n)(\partial n/\partial x)$. A linear analysis for perturbations varying as $\exp i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r})$ is performed on the set of moment equations[48] (i.e. conservation of particles, momentum and energy) for the separate electron and ion species, the set being closed by appropriate assumptions depending on whether two or three moments are used.

A variety of assumptions are common to the theories, namely the quasistatic one $(\underset{\sim}{E} = -\nabla\Phi)$, quasineutrality $(n_e = n_i)$, neglect of electron inertia, neglect of ion temperature fluctuations (ion energy equation omitted), and low frequencies $(\omega \ll \omega_{ci})$. The various theories also differ according to which of the following additional approximations are made:

(a) Neglect of electron energy fluctuations (isothermal theory, electron energy equation omitted)

(b) Neglect of ion parallel motion.

The various theories, the approximations made, and the order of the dispersion relation in ω (α) and $k_{\parallel}$ (β) are listed in Table I.

TABLE I

Approximations	Axial Current	Authors	α	β
a,b,	$v_{0\parallel} = 0$	Hendel, Chu and Politzer[46]; Rowberg and Wong[47]	2	2 (even)
a,	$v_{0\parallel} = 0$	Schlitt and Hendel[49]	3	4 (even)
b,	$v_{0\parallel} = 0$	Tsai, Perkins and Stix[50]	3	4 (even)
b,	$v_{0\parallel} \neq 0$	Tsai, Ellis and Perkins[51]	3	4 (odd)
	$v_{0\parallel} = 0$	Hartman and Watanabe[52]	4	6 (even)

In the earlier isothermal theories (Refs. 46, 47, 49) there is no effect due to the inclusion of axial current because of an exact cancellation when the electron continuity and momentum transfer equations are combined.† To incorporate this additional destabilizing effect it is necessary to treat T_e as a variable via the electron energy equation (Refs. 50-52). The dispersion relation then involves odd powers of k and the medium is non-reciprocal. The inclusion of parallel current in the theory of Ref. 52 would introduce odd powers of k but leave the order unchanged.

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† This exact cancellation only occurs for a fully ionized plasma. For weakly ionized plasmas, in which the charged particles collide with the neutral background, one obtains a destabilizing effect due to axial drift in an isothermal theory using the first two moment equations.[8,11]

(B) Correlation of Theory and Experiment for Collisional Drift Waves

(i) Isothermal Theory - No Axial Current

The earlier work[46,47] on collisional drift waves in symmetric (double-ended) Q-machines without axial current was interpreted in terms of the isothermal theory neglecting parallel ion motion. The dispersion relation can be written, with the convention $\exp i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{r})$:

$$D(w,K) \equiv w^2 + \left\{1 - i\left[\frac{(1+b)K^2}{b\,\nu_e} + C_i b\nu_i\right]\right\} w$$

$$-\left\{2\,\frac{C_i b\nu_i K^2}{\nu_e} - i\left[\frac{(1-b)K^2}{b\,\nu_e} - C_i b\nu_i\right]\right\} = 0 \qquad (19)$$

where

$$w = \omega/\omega_D \;;\quad \omega_D = k_y v_D \;;\quad v_D = \frac{-T\chi}{eB} \;;$$

$$\chi = (1/n)(\partial n/\partial x) \;;\quad T_e = T_i = T \text{ (energy units)}$$

$$b = \frac{k_\perp^2 T}{m_i \omega_{ci}^2} \;;\quad K^2 = \frac{k_\parallel^2 T}{\omega_D^2 m_e} \;;\quad \nu_i = \frac{\nu_{ii}}{\omega_D}$$

$$\nu_e = \frac{C_r \nu_{ei}}{\omega_D} \quad .$$

Here ν_{ei}, ν_{ii} are the collision frequencies as defined by Braginskii and the coefficient of electron parallel resistivity $C_r = 0.513$ for a singly ionized plasma. The coefficient of ion viscosity C_i is given as (3/10) by Braginskii but was taken as (1/4) in Refs. 46 and 47. With $C_i = 1/4$, Eq. (19) is the dispersion relation of Wong and Rowberg. That of Hendel et al differs slightly in terms involving b, which is necessarily small compared to unity for fluid theory.

In applying this theory to the cylindrical geometry of the Q-machine, k_y is real and determined experimentally by identifying it with m/a where m is the azimuthal mode number and a is the radius at which the azimuthally travelling wave has its maximum amplitude. Also k_x is determined from the radial wave profile to give

$k_\perp^2 = k_x^2 + k_y^2$. Furthermore $k_\parallel^2$ is determined from the axial profile of the wave, a matter we discuss at greater length below. All the other parameters are determined with reasonable precision from the experimental conditions.

A partial mapping of K into the w-plane via Eq. (19) for conditions corresponding to Rowberg and Wong's experiment is shown in Fig. 5. Because D is even in k , the real k-axis folds on itself and the two sheets are shown in a single diagram. The relevant branch point occurs at

$$w_s = iC_i b\nu_i \; ; \quad K_s = 0 \; .$$

Since $w_{si} > 0$ the system can never be absolutely unstable. The real k-axis terminates for $k = \pm\infty$ at $w = [(1-b) + i2C_i b\nu_i]/(1+b)$ which corresponds to stability in the limit of short parallel wavelength. At intermediate values of K , the real K-axis may or may not dip into the lower half plane, corresponding respectively to convective instability or stability, depending on the parameter values. The condition for convective instability is

$$8C_i^2 \frac{(1+b)}{(1-b)^2} b^3 \nu_i^2 < 1 \tag{20}$$

which for $b \ll 1$ may be approximated to

$$b^3 \lesssim (8C_i^2\nu_i^2)^{-1} \tag{20a}$$

This gives a threshold magnetic field for marginal instability $B_c \propto \nu_i^{1/3}$.

Correlation between theory and experiment has been made in one of three ways: comparison of the onset field and frequency for self-excited unstable modes;[46] comparison of the frequency and damping rate for externally excited modes when the system is stable;[47] comparison of the frequency and growth rate for self-excited modes when the system is unstable and feedback stabilization is switched off.[16] In all cases it is found experimentally that the system is not as unstable as theory predicts, i.e. the damping rates are larger or the growth rates smaller

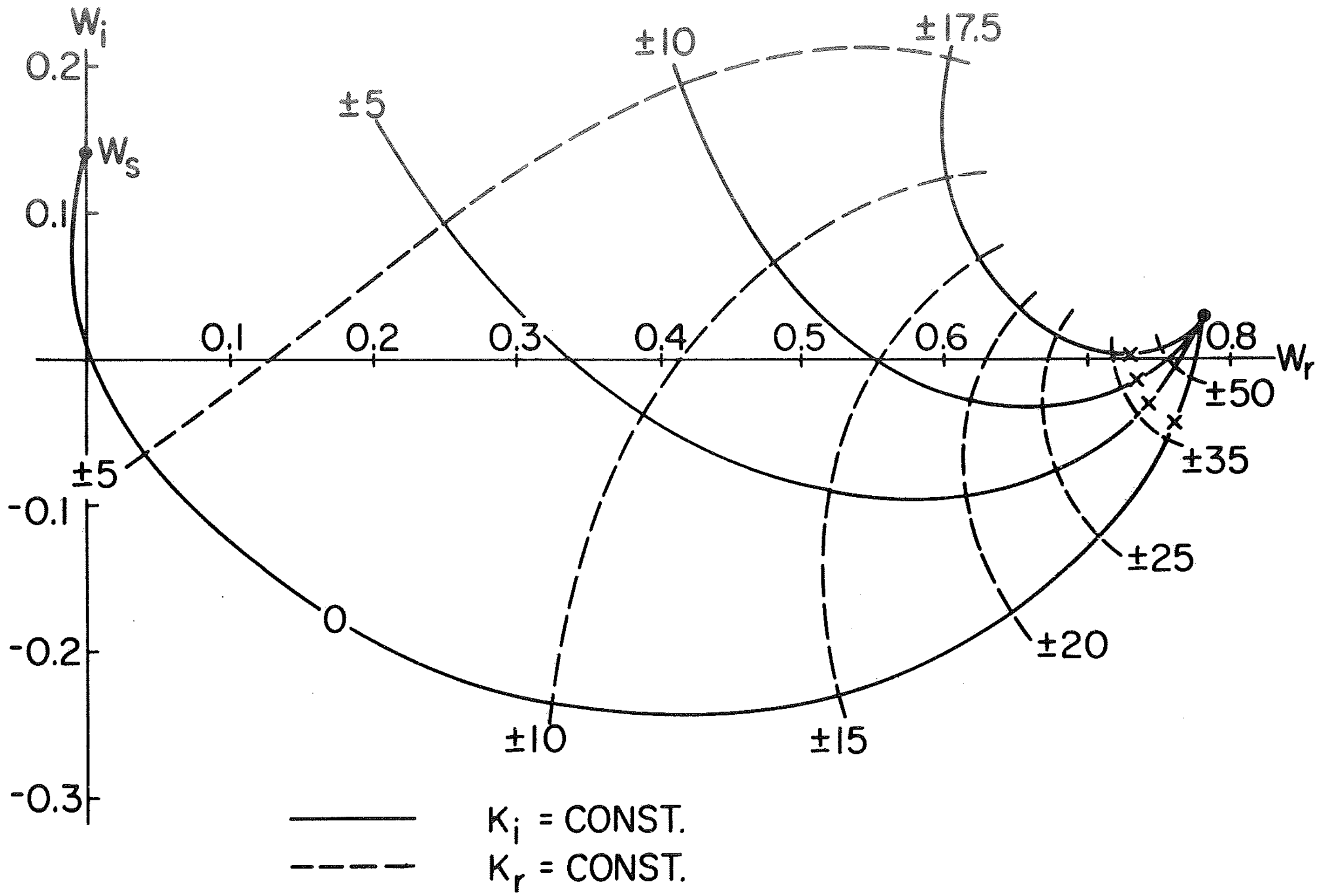

FIG. 5. PARTIAL MAP OF K INTO w-PLANE VIA EQ. (19) FOR CONDITIONS OF ROWBERG AND WONG'S EXPERIMENT: $m_i = 39.1$ AMU; $n = 6.7 \times 10^{10}$ cm^{-3}; $T = 0.19$ eV; $B = 1.6$ kG; $m = 2$; $k_y = k_x = 1.43^{1}$cm^{-1}; $\chi = -0.80$ cm^{-1}; $\nu_{ei} = 2.0 \times 10^7$ sec^{-1}; $\nu_{ii} = 5.2 \times 10^4$ sec^{-1}; $\omega_D = 1.36 \times 10^4$ sec^{-1}; $\omega_{ci} = 3.92 \times 10^5$ sec^{-1}; $\nu_e = 7.5 \times 10^2$; $\nu_i = 3.8$; $b = 0.124$. THE LOWEST AXIAL MODE ($\lambda_{\parallel} = 3.6$ L, L = 60 cm) IS SHOWN BY CROSSES.

than predicted. In terms of the onset field B_c for a given mode, the experimental value is a factor $\sim 1.5 - 2$ higher than predicted. There are several possible reasons for this discrepancy but the one which concerns us here is the possible effect of end-plate damping.

In practice it is usually found that, for a given m, the mode with the largest parallel wavelength $\lambda_\parallel (\equiv 2\pi/k_\parallel)$ is the most unstable, and most of the measurements have been restricted to this mode. This is to be expected from the shape of the map of the real $k_\parallel$-axis in the ω-plane (Fig. 5) unless the machine is so long that the lowest axial mode (smallest $k_\parallel$) lies well to the left of the minimum of $\omega(k_\parallel$ real), in which case the next mode could have a higher growth rate.

Hendel et al, and also Rowberg and Wong, measure an axial standing wave with $\lambda_\parallel \geq 2L$ for both electron and ion-rich sheaths. The former, who worked mostly with electron-rich sheaths found $\lambda_\parallel \approx 2L$ corresponding to a condition close to a short-circuit at the sheath edge. The latter, who worked mostly with ion-rich sheaths, investigated the dependence of $\lambda_\parallel$ on the potential U of the plasma relative to the end plates (Fig. 6). They found $\lambda_\parallel \approx 2L$ for an electron-rich sheath ($U = -0.05$ V) and $\lambda_\parallel \approx 3.6L$ for an ion-rich sheath ($U = +0.5$ V), in reasonable agreement with a formula due to Chen:[44]

$$\frac{k_\parallel L}{2} \tan \frac{k_\parallel L}{2} = \frac{L}{2\rho_i} \frac{\nu_{ei}}{\omega_{ci}} \left(\frac{m_e}{2\pi m_i}\right)^{1/2} \begin{Bmatrix} 1 \\ \exp{-eU/T} \end{Bmatrix} \quad \begin{matrix} U < 0 \\ U > 0 \end{matrix} \quad , \tag{21}$$

where ρ_i is the ion Larmor radius. For electron-rich sheaths, ($U < 0$), the r.h.s. of Eq. (21) is large and $\lambda_\parallel \approx 2L$ for the lowest mode. For ion-rich sheaths, ($U > 0$), the r.h.s. can become comparable or less than unity and $\lambda_\parallel > 2L$ for the lowest mode.

Equation (21) was derived[44] on the assumption that $k_\parallel$ is real and neglected any end-plate damping of the modes. Chen[44] discusses two damping processes connected with the end plates. The first is due to the fact that some charged particles, carrying wave energy, escape to the end plates and are replaced by new particles which do not have this wave energy. For ion-rich sheaths this applies to the ions and should

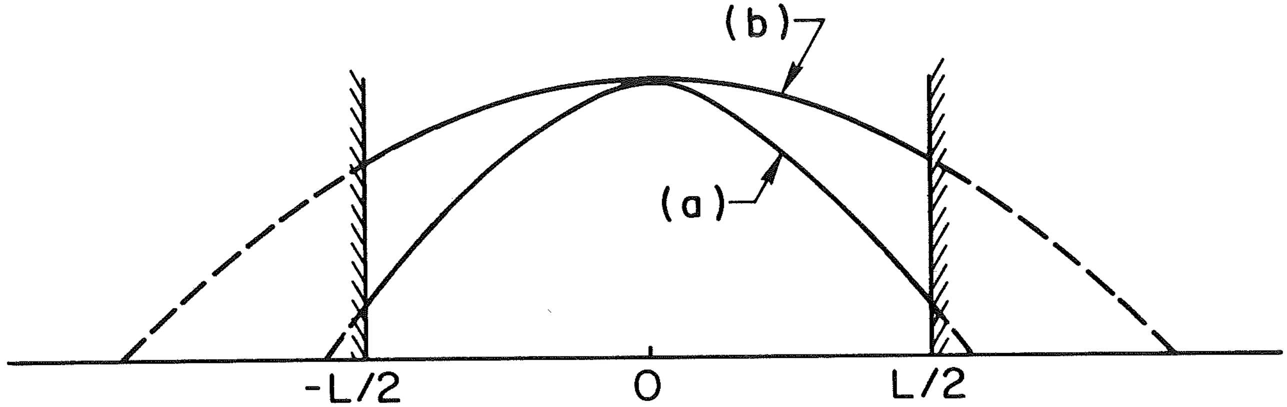

FIG. 6. AXIAL PROFILE OF WAVE AMPLITUDE FOR LOWEST AXIAL MODE

(a) ELECTRON-RICH SHEATHS, $\lambda_{\parallel} \approx 2L$

(b) ION-RICH SHEATHS, $\lambda_{\parallel} > 2L$

give rise to significant damping whereas for electron-rich sheaths it applies to the electrons and should give little damping. In either case, particles of the opposite charge are very largely reflected at the sheaths. The second process is end-plate diffusion resulting in randomization of the phase of ion gyration when ions are reflected at the end plates.

Rowberg and Wong[47] attribute the discrepancy between their observed and calculated growth rates for ion-rich sheaths to end-plate damping. Following Chen[44] they calculate a temporal damping decrement $\omega_{iD} \equiv (1/N_1)(\partial N_1/\partial t)$ where N_1 is the total number of perturbed ions in the column and $\partial N_1/\partial t$ is the rate at which such ions are lost by recombination at the end plates. For ion-rich sheaths the decrement is evaluated as

$$\omega_{iD} = \frac{2\, v_{ith}}{L} [1 + \exp e(U_w - U_I - U)/T]^{-1} \quad (U > 0) \qquad (22)$$

where v_{ith} is the ion thermal velocity, U_w is the end-plate work function and U_I the ionization potential of the neutrals. The experimental damping rate is then compared with the algebraic sum of the theoretical growth (damping) rate and the end-plate damping decrement calculated from Eq. (22).

While Eq. (22) is plausible and leads to values for ω_{iD} which agree approximately with the discrepancy between measured and calculated growth (decay) rates, its derivation can be criticized on several grounds. Firstly, the definition of N_1 as the number of perturbed ions is obscure; secondly, no account is taken of the spatial dependence of wave energy along the machine; thirdly, it is based on collisionless concepts and effectively distributes the damping uniformly along the system. It is clear from the viewpoint developed in the present paper that, within a fluid model, end-plate damping should be treated as a local effect producing imperfect wave reflection ($|\rho| < 1$). Consequently the mode is a partial standing wave constructed from roots $k^{\pm}(\omega)$ with complex $k^{\pm}$ corresponding to spatial growth towards the ends. The partial standing wave gives a flow of wave energy towards the ends where it is dissipated, and because of the spatial growth, the temporal growth rate is reduced.

Thus end-plate damping is incompatible with a pure standing wave ($k_{\parallel}^{\pm}$ real) as assumed by Rowberg and Wong. If the observed mode was indeed a pure standing wave then end-plate damping could not have been responsible for the discrepancy in temporal growth rate. On the other hand, if end-plate damping was operative, then the mode could not have been a pure standing wave. The theory of Sec. III should allow one to determine, from careful measurements of the axial mode pattern, the reflection coefficients and the values of $k_r^{\pm}$, $k_i^{\pm}$. Hence by solving the dispersion relation for these k values, the temporal growth rate can be determined, taking account of end-plate damping. Thus one can establish experimentally to what extent the discrepancy in temporal growth rates is attributable to end-plate damping.

Figure 7 shows curves of temporal growth rate ω_i versus magnetic field B, calculated from Eq. (19) for the conditions of Rowberg and Wong's experiment for the first axial mode with $m = 2$. The values of $|\rho| (=|\rho_1|=|\rho_2|)$ have been varied keeping the phase angle θ $(=\theta_1=\theta_2)$ fixed at $\pi/1.8$, to give $\lambda_{\parallel} \approx 3.6L$ when $|\rho| \approx 1$ as taken by Rowberg and Wong. It is seen that reducing $|\rho|$ reduces the growth rate and leads to damping, but it is clear that no single $|\rho| < 1$ curve matches the experimental results. The $|\rho| = 0.3$ curve matches at low B but has the wrong shape at large B. On the other hand, Rowberg and Wong's procedure of subtracting a constant damping decrement $\omega_{iD} \approx 1.4 \times 10^3$ sec^{-1} from the theoretical curve for real $k_{\parallel}$ ($|\rho| = 1$) does lead to a reasonable match with the experiment. However, in the experiment $\lambda_{\parallel}$ is sufficiently small that the effect of ion parallel motion is important, especially at higher B, so we should not expect agreement with a theory neglecting ion parallel motion.

Including ion parallel motion in the isothermal theory leads to the dispersion relation:

$$w^3 + \alpha_2 w^2 + \alpha_1 w + \alpha_0 = 0 \qquad (23)$$

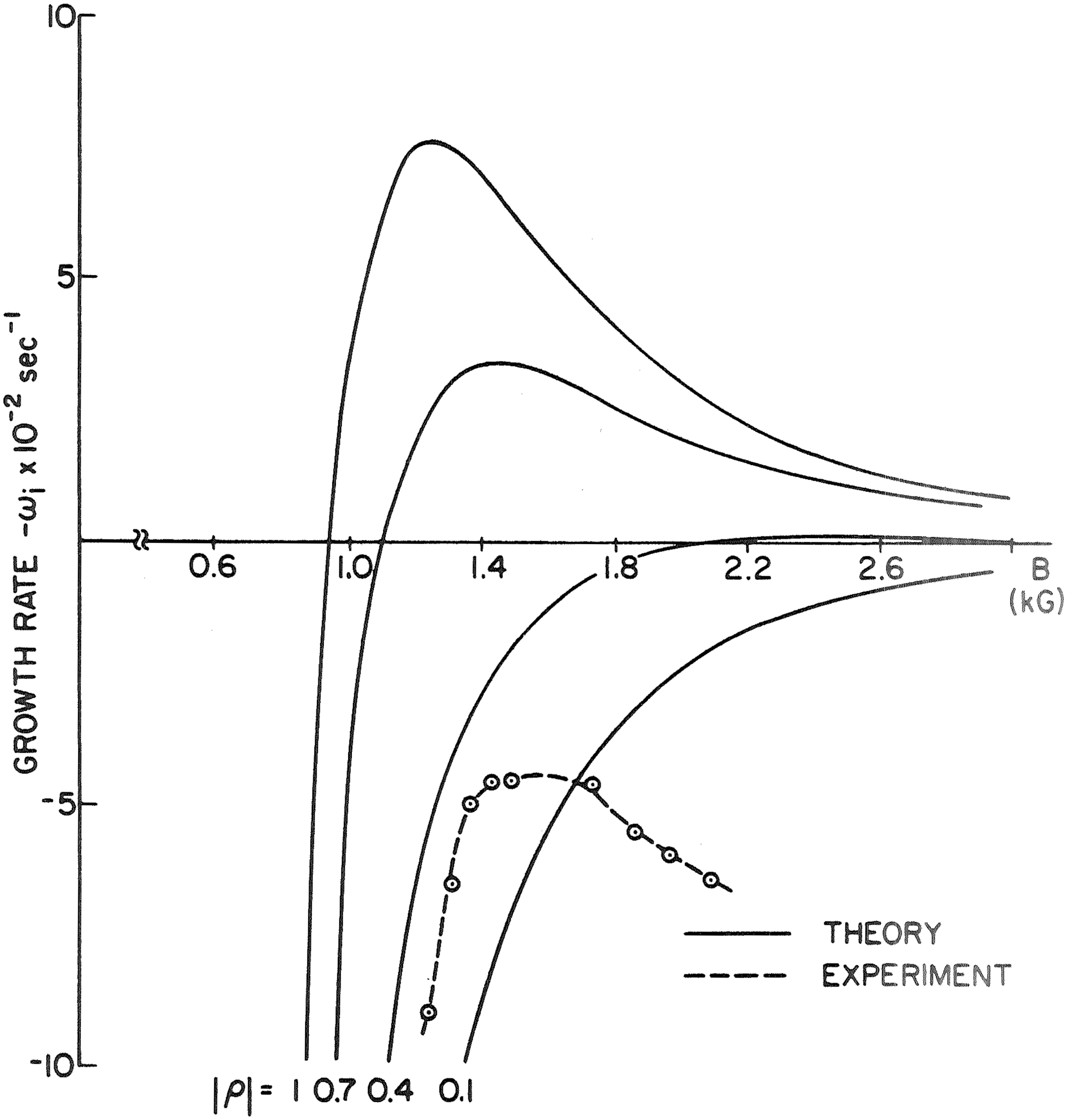

FIG. 7. GROWTH RATES AS A FUNCTION OF MAGNETIC FIELD CALCULATED FROM EQ. (19) FOR CONDITIONS OF ROWBERG AND WONG'S EXPERIMENT FOR VARIOUS END-PLATE REFLECTION COEFFICIENTS, COMPARED WITH EXPERIMENTAL VALUES.

where

$$\alpha_2 \equiv 1 - i \left[\frac{(1+b)K^2}{b\nu_e} + 5C_i b\nu_i \right]$$

$$\alpha_1 \equiv - \left[\left(\frac{4}{b} + 6 \right) \frac{K^2 C_i b\nu_i}{\nu_e} + 4 \left(C_i b\nu_i \right)^2 \right] + i \left[\frac{(1-b)}{b} \frac{K^2}{\nu_e} - 5C_i b\nu_i \right]$$

$$\alpha_0 \equiv \left[\frac{4c_i \nu_i (1-b)K^2}{\nu_e} - 4 (c_i b\nu_i)^2 \right]$$

$$+ i \left[\left(\frac{8c_i b\nu_i}{\nu_e} + 2 \frac{m_e}{m_i} \right) c_i b\nu_i K^2 + 2 \frac{m_e}{m_i} \frac{(1-b)}{b\nu_e} K^4 \right] .$$

Figure 8 shows the growth rate as a function of B calculated from Eq. (23) for various values of $|\rho|$ for the same conditions as Fig. 7. Comparing Figs. 7 and 8 it is seen that the inclusion of ion parallel motion reduces the growth rate appreciably, especially at large B . As a result there is now much better agreement between the experimental damping curve and a theoretical one taking account of end-plate damping through a reflection coefficient $|\rho| < 1$. The best match occurs for $|\rho| \approx 0.6$. The relatively poor match for low B is probably due to the fact that b $(\propto B^{-2})$ is not particularly small.

Figure 9 shows the amplitude and phase of the lowest axial mode for various values of $|\rho|$. For $|\rho| = 0.6$ the phase shift over the central region accessible to measurements is no more than 6°. Now, in stating that the observed axial mode was a standing wave, Rowberg and Wong placed no experimental limit on how small the phase shift was. However, it seems probable that a phase shift of 6° went undetected, especially in view of the fact that the phase varies rapidly with azimuth (m times the angle), so that detection of a small axial shift presupposes a rather precise tracking of the probe along a field line.

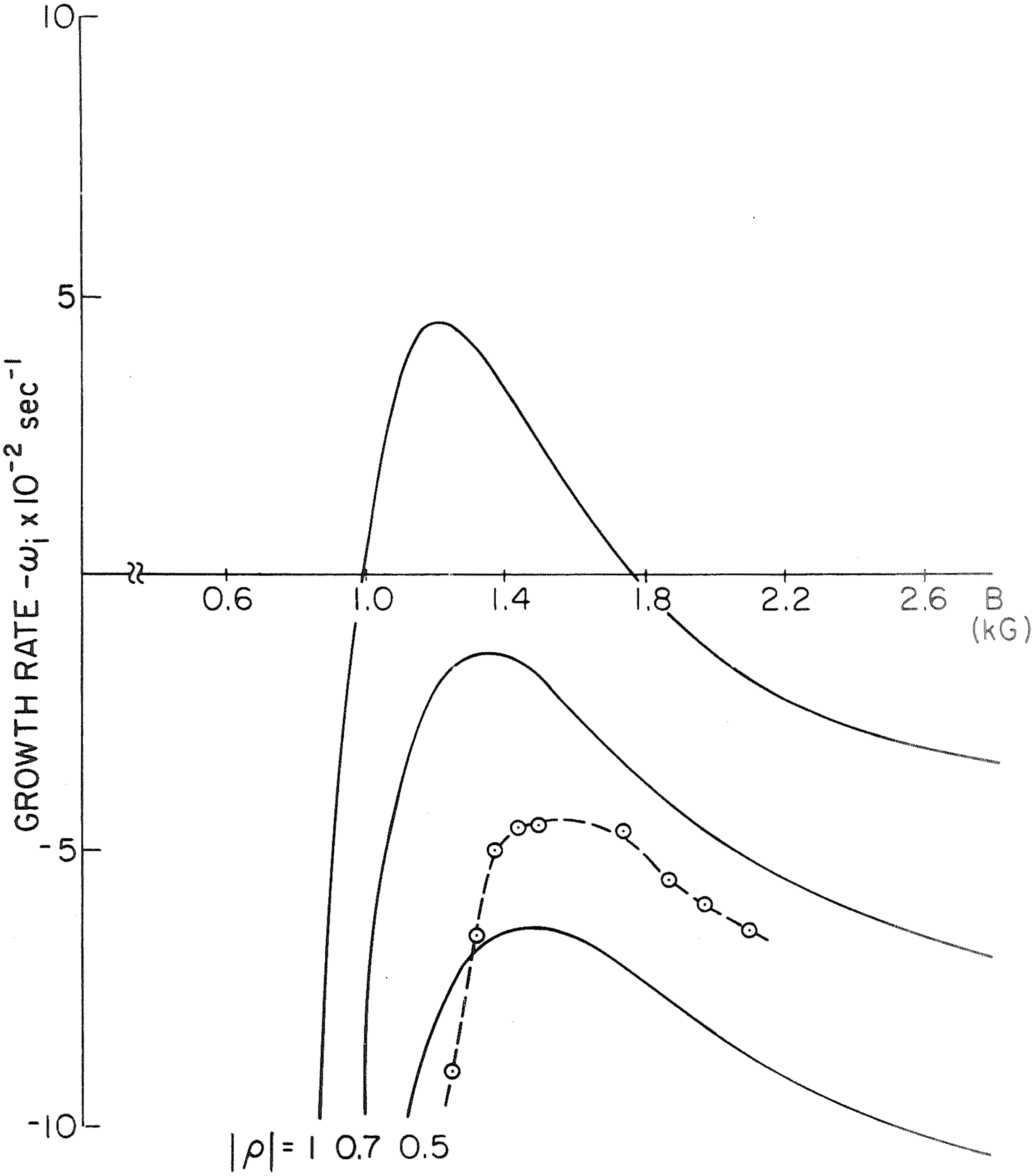

FIG. 8. GROWTH RATES AS A FUNCTION OF MAGNETIC FIELD CALCULATED FROM EQ. (23) FOR CONDITIONS OF ROWBERG AND WONG'S EXPERIMENT FOR VARIOUS END-PLATE REFLECTION COEFFICIENTS, COMPARED WITH EXPERIMENTAL VALUES.

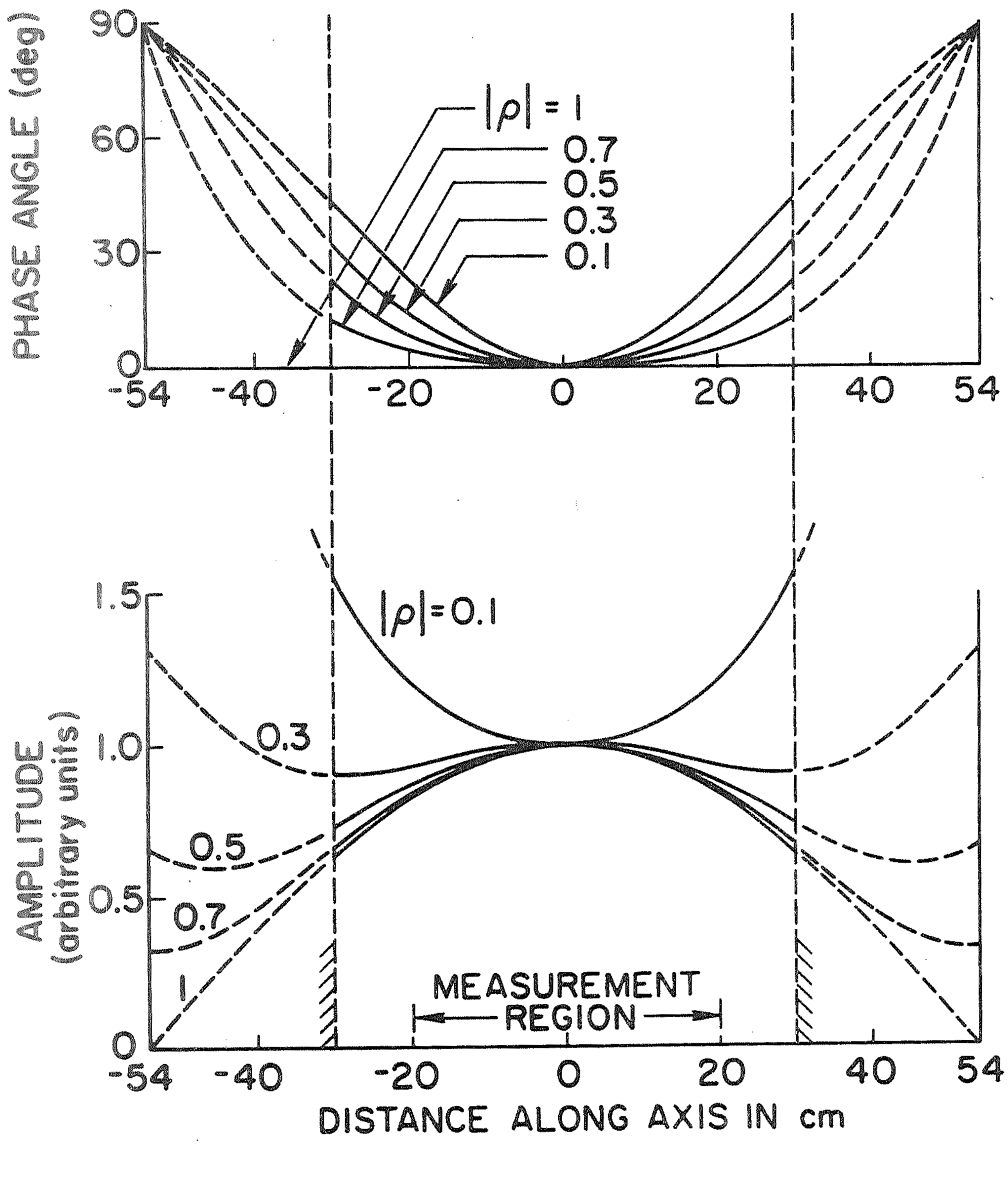

FIG. 9. AMPLITUDE AND PHASE OF THE AXIAL MODE AS A FUNCTION OF POSITION FOR VARIOUS REFLECTION COEFFICIENTS $|\rho|$ KEEPING THE PHASE θ CONSTANT.

Thus, from the published data it is difficult to assess to what extent the discrepancy in growth rate is attributable to end-plate damping, but more precise measurements of axial phase shift should enable the fact to be established. In the case of the experiments of Hendel et al with electron-rich sheaths, the effect of end-plate damping was probably smaller. However, as is clear from the results of Figs. 7-9, end-plate damping can give rise to appreciable increase in the threshold magnetic field and yet give little detectable axial phase shift.

Finally, it may be noted that the difficulty of detecting the presence of end-plate damping from the axial mode structure is greatest for the lowest axial modes $\lambda_{\parallel} \geq 2L$ because there is no minimum of the wave amplitude in the system. If measurements could be made around the minimum of a higher-order axial mode the reflection coefficient could be determined with greater accuracy.

(ii) Non-Isothermal Theory and Effect of Axial Current

A priori, there is no reason to believe that collisional drift waves are isothermal. It is just that, within a fluid treatment, the simplest theory results from using the first two moment equations, closing the set by assuming constant T. By treating T as a wave variable, and using a scalar third moment (energy) equation to complete the set, one expects the pressure waves obtained from the isothermal theory to be modified and new wave types (temperature waves) to appear, since the dispersion relation is of higher order. For weakly ionized plasmas it is well known that a non-isothermal theory is necessary to explain striations. In that case, small temperature fluctuations produce large changes in ionization rate and the temperature waves become unstable ionization waves.[53] Temperature fluctuations have also been shown to be important in drift-type instabilities in P.I.G. discharges.[54]

The isothermal theory of collisional drift waves, neglecting ion parallel motion, has been generalized by Tsai, Perkins and Stix[50] to include T_e as a variable. They find that the drift waves are modified so that the discrepancy in onset field is reduced. They also find a new root $\omega(k)$ which they label an entropy wave. This is stable under conditions that the drift wave is usually observed, but can be unstable under other conditions. Experimental evidence for the fact that drift waves are indeed non-isothermal has been provided by Motley and Ellis[55] from a careful interpretation of Langmuir probe measurements.

The non-isothermal theory of Ref. 50 has been generalized by Tsai, Ellis and Perkins[51] to include the effect of an axial current (electron velocity $v_{0\parallel}$). This introduces odd powers of k into the dispersion relation so that the medium is non-reciprocal with respect to propagation in the $\pm z$ directions. Neglecting zero-order temperature gradients and taking $T_i = T_e$, the dispersion relation can be written in the form:

$$D(w,K) \equiv w^3 + \beta_2 w^2 + \beta_1 w + \beta_0 = 0 \tag{24}$$

where

$$\beta_2 \equiv \left[1 - C_1 Kv\right] - i\left[\left(\frac{1+b}{b} + C_2\right)\frac{K^2}{\nu_e} + C_i b\nu_i\right]$$

$$\beta_1 \equiv -\left[\frac{(1+b)}{b}\,\frac{(C_2-C_3)K^4}{\nu_e^2} + \left\{\frac{(C_2+2)C_i b\nu_i}{\nu_e} + \frac{(2+b)v^2}{b}\right\}K^2 + C_1 Kv\right]$$

$$+\, i\left[\frac{(C_1-C_4)b + (C_1-C_5)}{b\nu_e}\,K^3 v + \left(\frac{1-b}{b} - C_2\right)\frac{K^2}{\nu_e} + C_1 C_i b\nu_i Kv - C_i b\nu_i\right]$$

$$\beta_0 \equiv \left[\frac{(1-b)}{b}\,\frac{(C_2-C_3)}{\nu_e^2}\,K^4 + \frac{(2C_1-C_4-C_5)C_i b\nu_i K^3 v}{\nu_e} + \left\{\frac{(1-b)}{b}\,v^2 - \frac{C_2 C_i b\nu_i}{\nu_e}\right\}K^2\right]$$

$$+\, i\left[2\,\frac{(C_2-C_3)}{\nu_e^2}\,C_i b\nu_i K^4 - \frac{(1-b)}{b}\,\frac{(C_1-C_4)}{\nu_e}\,K^3 v + 3C_i b\nu_i K^2 v^2 + C_1 C_i b\nu_i Kv\right]$$

where $c_1 \equiv \frac{2}{3}\left(\frac{7}{2} c_t + 5\right) \approx 5.00$,

$$c_2 \equiv \frac{2}{3} [c_e c_r + (c_t + 1)^2] \approx 3.03 ,$$

$$c_3 \equiv \frac{2}{3} (c_t + 1)^2 \approx 1.95 ,$$

$$c_4 \equiv \frac{5}{3} (c_t + 1) \approx 2.85 ,$$

$$c_5 \equiv \frac{7}{3} (c_t + 1) \approx 3.99 ,$$

$c_t = 0.711$, $c_e = 3.16$, $c_r = 0.513$, and $v = v_{0\parallel}(T/m_e)^{-1/2}$. Equation (24) differs slightly from that of Ref. 51 in terms involving b because no approximations involving $b \ll 1$ have been made.

Figure 10 shows the map of the real K-axis into the w-plane via Eq. (24) with $v = 0$ for the same parameters as Figs. 5, 7 ~~and~~ 8. The broken lines show the same contour from the isothermal theory, Eq. (19). Branch I (the drift wave) is strongly modified at small K , and shows lower growth rate for $K \lesssim 20$, but higher growth rate for $K \gtrsim 20$, corresponding to Rowberg and Wong's short machine. There is also a significant change in frequency, w_r . Unfortunately the frequency is not a good datum for distinguishing the presence of non-isothermal effects in experiments because, to compare with the slab model theory, one has to correct for the azimuthal Hall velocity $v_\theta = E_r/B_z$ from measurements of E_r , which are not very precise. Branch II (labelled by Tsai et al as the flute mode) is little affected by the inclusion of non-isothermal effects, while Branch III is the new branch (entropy wave) which is weakly unstable for very long wavelengths.

Figure 11 shows the real K-axis mapped into the w-plane for the same parameters as Fig. 10 but with $v = 1.86 \times 10^{-2}$, corresponding, for potassium ions, to $v_{0\parallel} = 5\, c_s$, where $c_s \equiv (T/m_i)^{1/2}$ is the ion sound speed. The case $v = 0$ (from Fig. 10) is shown by a broken line for comparison. When $v \neq 0$, the real K-axis no longer folds on itself and the lowest relevant branch point has migrated from $w_s = 0$ to $w_s \approx (-0.13 - i\, 0.085)$ so the infinite system is absolutely unstable. The drift wave (I) and entropy wave (III) branches are strongly modified

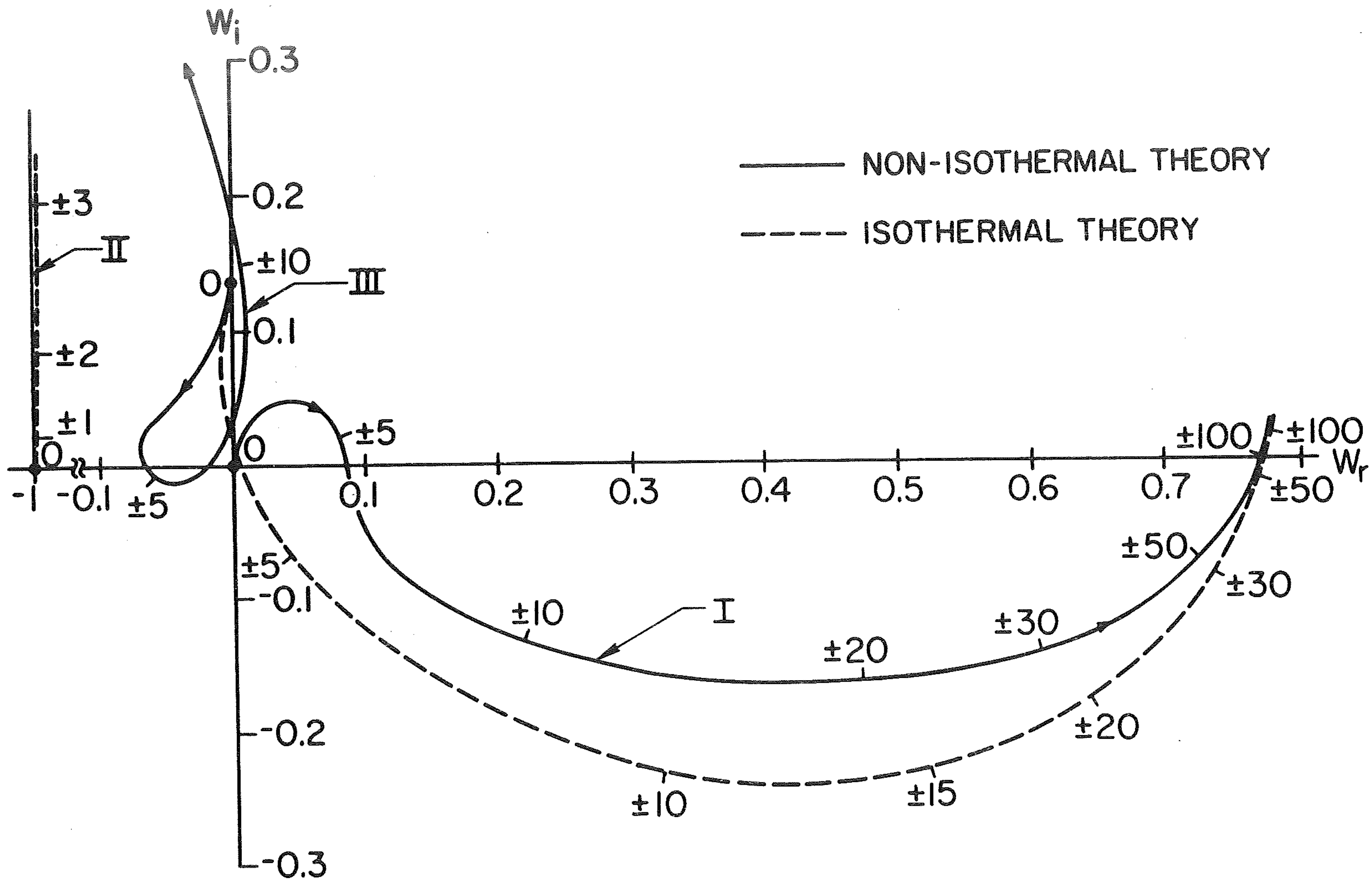

FIG. 10. MAP OF THE REAL K-AXIS INTO THE w-PLANE FOR NONISOTHERMAL THEORY (EQ. 24) COMPARED WITH THAT FOR ISOTHERMAL THEORY (EQ. 19). PARAMETER VALUES ARE AS IN FIG. 5.

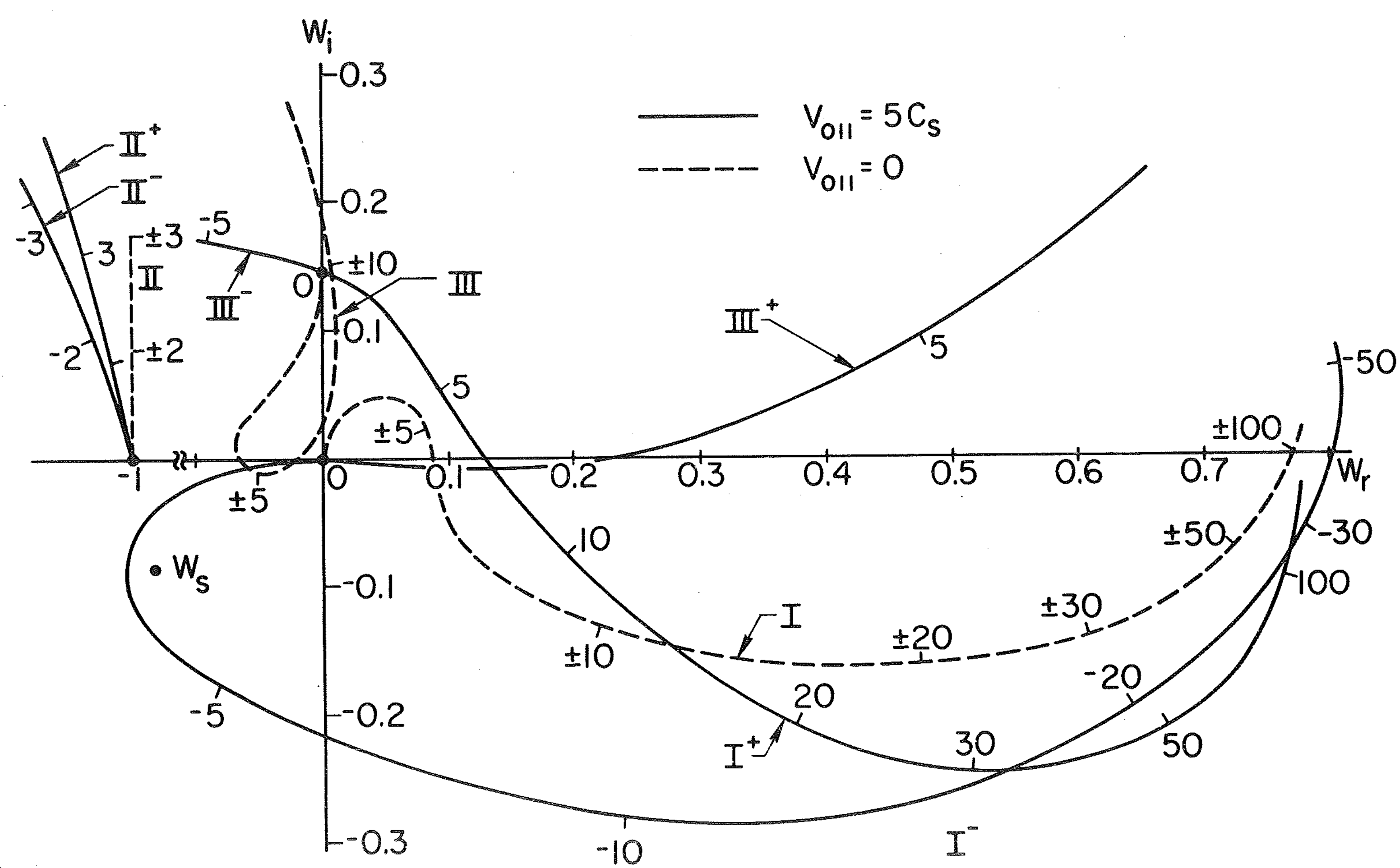

FIG. 11. MAP OF THE REAL K-AXIS INTO w-PLANE FOR NON-ISOTHERMAL THEORY (EQ. 24) WITH AXIAL CURRENT ($v = 1.86 \times 10^{-2}$) COMPARED WITH CURRENT FREE CASE. PARAMETER VALUES AS IN FIGS. 5 and 10.

by the current, especially for small $|K|$. In fact the topology is changed so that the drift ($I^{\pm}$) and entropy wave ($III^{\pm}$) branches are interconnected. For $|K| \gtrsim 10$, where the branches $I^{\pm}$ can be identified as drift waves (I) modified by the current, the current is destabilizing for negatively directed waves for $K \gtrsim -22$ and stabilizing for $K \lesssim -22$. On the other hand, for the positively directed wave, the current is stabilizing for $K \lesssim 15$ and destabilizing for $K \gtrsim 15$. Thus for large $|K|$ the effect of the current is destabilizing or stabilizing according as the phase velocity is parallel or antiparallel to the electron drift. However, at large $|K|$ it is probably necessary to include the effect of ion parallel motion as discussed by Schlitt and Hendel[49] in the isothermal theory.

Tsai et al[51] consider the boundary conditions for a Q-machine and the construction of normal axial modes. The boundary conditions are derived only in the limit of strongly electron-rich sheaths and neglect end-plate damping. In these limits they obtain the conditions

$$n_i/n_0 = e\Phi_1/T_{e0} \tag{25a}$$

and

$$T_{e1}/T_{e0} = 0 \quad . \tag{26a}$$

Condition (25a) agrees approximately with Chen[46] in the limit of electron-rich sheaths and corresponds to a short circuit. Tsai et al then construct normal axial modes from the four roots $k_{\parallel}(\omega)$ satisfying these boundary conditions. For conditions of interest, two roots (labelled drift waves) are dominant while the other two (labelled entropy waves) are heavily damped and only have significant amplitude near the end plates. Because the medium is non-reciprocal they find the modes are partial standing waves, even though no end-plate damping is assumed.

To discuss more generally the dependence of the normal axial modes on the terminations, when these are specified in terms of reflection coefficients for the dominant waves, a map such as Fig. 11 is not convenient. A convenient procedure is to map into the w-plane contours of constant $(K_r^+ - K_r^-)$ and $(K_i^+ - K_i^-)$ for the dominant modes as shown in

Fig. 12 for the same parameters as Fig. 11. If values of ρ_1, ρ_2 for these dominant modes are given, either from measurements or theory, the difference $(K^+ - K^-)$ can be calculated from Eq. (9) and the normal mode frequencies read off from Fig. 12. The contour $(K_i^+ - K_i^-) = 0$ is the locus of normal mode frequencies when the terminations are perfectly reflecting. Comparing this with the broken line for the case $v = 0$ and perfect reflections, it is seen that the current destabilizes the normal axial modes over a wide range of $(K_r^+ - K_r^-)$ values.

(c) General Remarks

While the approximate agreement between experiment and the earlier isothermal theory neglecting ion parallel motion was good by accepted standards in plasma physics, it is clearly important to identify and take account of those factors in the theory necessary to produce more detailed agreement. Two of these factors, electron temperature variation and ion parallel motion have been identified. Another factor, whose effect is difficult to assess quantitatively is the lack of satisfaction of the local approximation $k_x \gg \chi$ and the associated condition $k_y \gg k_x$ employed in the slab model.

In addition to these factors associated with the theory for an infinitely long system, we have emphasized the need to consider the normal axial mode structure for a finite-length system, and show how these may be constructed when the terminations are specified in terms of reflection coefficients for the dominant waves. It was shown that a reflection loss which is small enough to give a relatively small detectable effect on the axial mode structure, compared with the lossless case, may give a significant reduction in temporal growth rate and a rise in the instability threshold. Furthermore, the recognition that end-plate damping implies a partial standing wave, even in the case of zero axial current, leads to an experimental method of determining its magnitude in terms of a reflection coefficient $|\rho| < 1$, analogous to to the methods used for arbitrary terminations in passive transmission lines.

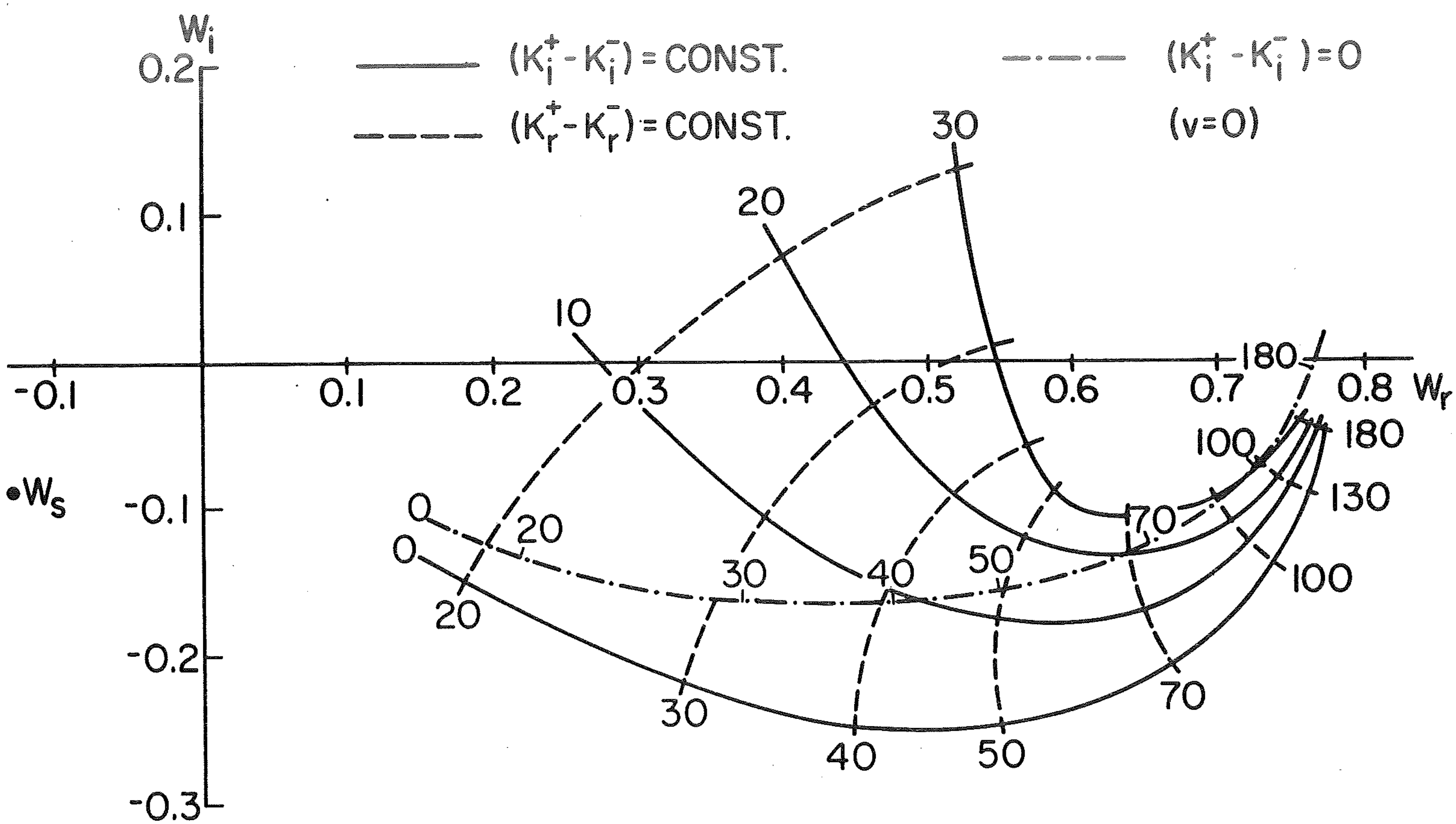

FIG. 12. MAP OF THE DIFFERENCE $(K^+ - K^-)$ INTO THE w-PLANE FOR THE DOMINANT (DRIFT) WAVES FOR THE CASE OF FIG. 11 ($v = 1.86 \times 10^{-2}$). THE REAL $(K^+ - K^-)$ CONTOUR IN THE CASE $v = 0$ IS SHOWN AS A CHAIN-DOT LINE.

In a final analysis it should be possible to relate these experimental reflection coefficients to a suitably detailed physical theory of the end plates. The present theories for the end plates are approximate and incomplete. For instance, if both electron temperature variations and ion parallel motion were included one would need three boundary conditions on the wave variables at each boundary to construct complete normal modes from the six roots $k(\omega)$. In the absence of such a detailed theory, the approach suggested here leads to a practical method of determining the influence of the terminations on the axial modes.

V. DISCUSSION

In this paper we have discussed the general question of how, and under what conditions, it is possible to construct normal modes for an arbitrarily bounded system from roots of the dispersion relation $D(\omega,\underset{\sim}{k}) = 0$ for a corresponding unbounded system. The treatment is restricted to systems described by fluid-type equations which are uniform along one coordinate. With these restrictions, normal modes may be constructed from a pair of dominant roots $k^{\pm}(\omega)$ of $D = 0$, where the allowed (complex) values of the difference $(k^+ - k^-)_n$ are given by Eq. (9) in terms of the boundary conditions, expressed as reflection coefficients for these dominant waves. The corresponding (complex) normal mode frequencies, ω_n, are given by Eq. (12). The solutions only represent normal modes if $\text{Im}\,\omega_n < \text{Im}\,\omega_s$, where (ω_s,k_s) is the lowest (dominant) relevant branch/saddle point of $D = 0$ in the sense of Derfler and Briggs. The axial structure of the normal modes is a partial standing wave described by Eq. (14) with Eqs. (17) and (18).

A number of general conclusions were drawn concerning how the system behavior depends on the properties of $D = 0$ and the values of the reflection coefficients $\rho_{1,2}$. In general, provided the terminations are sufficiently loss-free ($|\rho_{1,2}|$ not too small), normal modes are found whether $D = 0$ represents stability, convective instability or absolute instability. In the latter cases, temporally growing normal modes are found provided the system length is appropriate. As the terminations are made more lossy, ($|\rho_{1,2}|$ reduced), temporally growing modes are stabilized in the case $D = 0$ represents convective instability, but the system still supports spatially growing waves excited for real frequency. When $D = 0$ represents absolute instability, increased end losses reduces the growth rates of normal modes until they all grow less rapidly than waves associated with the lowest relevant branch/saddle point (ω_s,k_s). In the limit of non-reflecting boundaries, $|\rho_{1,2}| = 0$, the system is effectively infinite and a localized perturbation grows as described asymptotically by Eq. (5).

In general the normal axial modes are partial standing waves composed of travelling waves with distinct complex values $k^{\pm}$. Only when the system is reciprocal (D even in k, so $k^+ = -k^-$) and the terminations purely reactive ($|\rho_{1,2}| = 1$) are the component $k^\pm$ values purely real. For non-reciprocal systems, such as plasma columns with electrons drifting axially through ions, the normal modes in general have $k^\pm$ complex even when the terminations are lossless.

In Sec. IV these ideas were applied to collisional drift waves in Q-machines. It was shown that the discrepancy between the observed and calculated growth rates in Rowberg and Wong's experiment may be explained by end-plate damping, but not in the manner they suggest. Only if the mode is a partial standing wave and account is taken of axial ion motion can the discrepancy be explained by end-plate damping. It was shown that end-plate damping can have a marked stabilizing effect, and yet give little detectable axial phase shift (on the commonly observed lowest axial mode). The dependence of the normal mode frequencies, growth rates and axial structure on the reflection coefficients was also discussed for the case of a Q-machine with axial current.

By characterizing the terminations by reflection coefficients for the dominant waves, determined by measurements on the partial standing wave at large distances from the boundaries, a number of factors are absorbed to which it is difficult to give a detailed theoretical treatment. These include effects due to the excitation of other wave types, including surface waves, sheath effects, and the transition from a fluid description to a kinetic description within a mean free path of the boundary.

Perhaps the most important point to which we have drawn attention, is that the behavior of practical (bounded) systems, with respect to linear perturbations, is not generally describable in terms of roots $\omega(\underset{\sim}{k}$ real) of the dispersion relation for an unbounded or periodically bounded system, as is often assumed. To be sure, such a description is appropriate when the planar model is adapted to toroidal geometry and periodicity conditions allow travelling wave solutions with real $k_\parallel$. Furthermore, a description in terms of a pure standing wave (real $k^\pm$ waves) is appropriate for reciprocal systems with purely reactive terminations. However, in general, bounded systems require a description in terms of roots $k(\omega)$ of $D = 0$ with complex k. The requirement that one, at least, of the waves be spatially growing results in lower temporal growth rates and can result in a quiescent system even when the infinite system dispersion relation indicates (convective) instability. It seems probable that such effects due to boundaries explain[56] why, in mirror machines, certain predicted instabilities are not observed.

The foregoing observations also imply that in practice many nonlinear wave phenomena (e.g. inhomogeneous turbulence) cannot be adequately explained by mode coupling theories based on the coupling of linearly independent modes $\omega(\underset{\sim}{k}$ real) of the linear theory.

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